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Analytic Geometry and the Conic Sections

CHAPTER OUTLINE

- 10.1** A Brief Introduction to Analytical Geometry 920
- 10.2** The Circle and the Ellipse 927
- 10.3** The Hyperbola 940
- 10.4** The Analytic Parabola 954
- 10.5** Polar Coordinates, Equations, and Graphs 965
- 10.6** More on the Conic Sections: Rotation of Axes and Polar Form 978
- 10.7** Parametric Equations and Graphs 995

CHAPTER CONNECTIONS

One of the most breath-taking, dare devil stunts performed at air shows is the *power dive*. In some cases, as the pilot dives toward the ground and pulls out of the dive just in time, the plane flies along a path that can be modeled by a hyperbola (the third member of the family of conic sections). If we consider a given point on the ground and under the grandstand as the origin $(0, 0)$, we can use the equation that models the hyperbolic path of the plane to determine its minimum altitude as it passes over the stands. This application appears as Exercise 81 in Section 10.3.

Check out these other real-world connections:

- ▶ Designing an Elliptical Garden (Section 10.2, Exercise 66)
- ▶ The Design of a Lithotripter for treating kidney stones (Section 10.2, Exercise 68)
- ▶ Locating a Ship Using Radar (Section 10.3, Exercise 85)
- ▶ Parabolic Shape of a Solar Furnace (Section 10.4, Exercise 90)

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10.1 A Brief Introduction to Analytical Geometry

Learning Objectives

In Section 10.1 you will learn how to:

- A.** Verify theorems from basic geometry involving the distance between two points
- B.** Verify that points (x, y) are an equal distance from a given point and a given line
- C.** Use the defining characteristics of a conic section to find its equation

Generally speaking, **analytical geometry** is a study of geometry using the tools of algebra and a coordinate system. These tools include the midpoint and distance formulas; the algebra of parallel, perpendicular, and intersecting lines; and other tools that help establish geometric concepts. In this section, we'll use these tools to verify certain relationships, then use these relationships to introduce a family of curves known as the **conic sections**.

A. Verifying Relationships from Plane Geometry

For the most part, the algebraic tools used in this study were introduced in previous chapters. As the midpoint and distance formulas play a central role, they are restated here for convenience.

Algebraic Tools Used in Analytical Geometry

Given two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in the xy -plane.

Midpoint Formula
The midpoint of line segment P_1P_2 is

$$(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Distance Formula
The distance from P_1 to P_2 is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

These formulas can be used to verify the conclusion of many theorems from Euclidean geometry, while providing important links to an understanding of the conic sections.

EXAMPLE 1 ▶ Verifying a Theorem from Basic Geometry

A theorem from basic geometry states: *The midpoint of the hypotenuse of a right triangle is an equal distance from all three vertices.* Verify this statement for the right triangle formed by $(-4, -2)$, $(4, -2)$, and $(4, 4)$.

Solution ▶ After the plotting points and drawing a triangle, we note the hypotenuse has endpoints $(-4, -2)$ and $(4, 4)$, with midpoint

$$\left(\frac{4 + (-4)}{2}, \frac{4 + (-2)}{2} \right) = (0, 1).$$

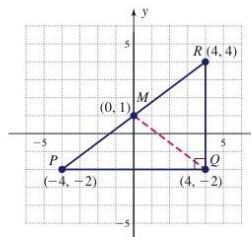
Using the distance formula to find the distance from $(0, 1)$ to $(4, 4)$ gives

$$\begin{aligned} d &= \sqrt{(0 - 4)^2 + (1 - 4)^2} \\ &= \sqrt{(-4)^2 + (-3)^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

From the definition of midpoint, $(0, 1)$ is also 5 units from $(-4, -2)$. Checking the distance from $(0, 1)$ to the vertex $(4, -2)$ gives

$$\begin{aligned} d &= \sqrt{(4 - 0)^2 + (-2 - 1)^2} \\ &= \sqrt{4^2 + (-3)^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

The midpoint of the hypotenuse is an equal distance from all three vertices (see the figure).



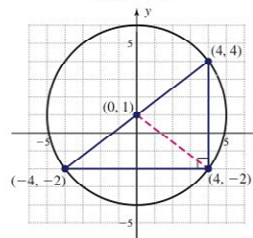
Now try Exercises 7 through 12 ▶

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Recall from Section 2.1 that a circle is the set of all points that are an equal distance (called the radius) from a given point (called the center). If all three vertices of a triangle lie on the circumference of a circle, we say the circle **circumscribes** the triangle. Based on our earlier work, it appears we could also state the theorem in Example 1 as *For any circle in the xy -plane whose center (h, k) is the midpoint of the hypotenuse of a right triangle, all vertices lie on the circle defined by $(x - h)^2 + (y - k)^2 = (\frac{d}{2})^2$, where d is the length of the hypotenuse.* See Figure 10.1 and Exercises 13 through 20.

A. You've just learned how to verify theorems from basic geometry involving the distance between two points

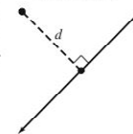
Figure 10.1



B. The Distance between a Point and a Line

In a study of analytical geometry, we are also interested in the distance d between a point and a line. This is always defined as the **perpendicular distance**, or the length of a line segment perpendicular to the given line, with the given point and the point of intersection as endpoints (see Figure 10.2).

Figure 10.2



EXAMPLE 2 ▶ Locating Points That Are an Equal Distance from a Given Point and Line

In Figure 10.3, the origin $(0, 0)$ is seen to be an equal distance from the point $(0, 2)$ and the line $y = -2$. Show that the following points are also an equal distance from $(0, 2)$ and $y = -2$:

- a. $(2, \frac{1}{2})$ b. $(4, 2)$ c. $(8, 8)$

Solution ▶ Since the given line is horizontal, the perpendicular distance from the line to each point can be found by vertically counting the units. It remains to show that this is also the distance from the given point to $(0, 2)$ (see Figure 10.4).

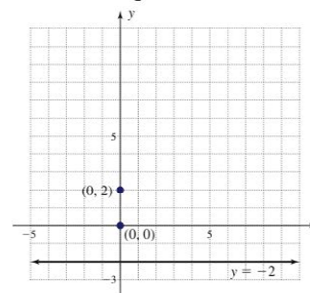
- a. The distance from $(2, \frac{1}{2})$ to $y = -2$ is **2.5 units**. The distance from $(2, \frac{1}{2})$ to $(0, 2)$ is

$$\begin{aligned} d &= \sqrt{(0 - 2)^2 + (2 - 0.5)^2} \\ &= \sqrt{(-2)^2 + (1.5)^2} \\ &= \sqrt{6.25} \\ &= 2.5 \quad \checkmark \end{aligned}$$

- b. The distance from $(4, 2)$ to $y = -2$ is **4 units**. The distance from $(4, 2)$ to $(0, 2)$ is

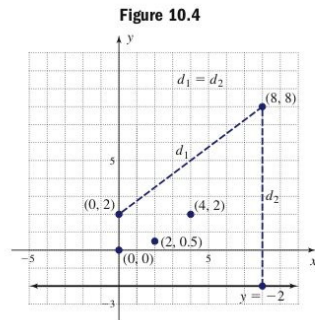
$$\begin{aligned} d &= \sqrt{(0 - 4)^2 + (2 - 2)^2} \\ &= \sqrt{(-4)^2 + (0)^2} \\ &= \sqrt{16} \\ &= 4 \quad \checkmark \end{aligned}$$

Figure 10.3



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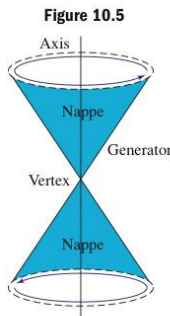
- c. The distance from (8, 8) to $y = -2$ is **10 units**. The distance from (8, 8) to (0, 2) is
- $$d = \sqrt{(0 - 8)^2 + (2 - 8)^2}$$
- $$= \sqrt{(-8)^2 + (-6)^2}$$
- $$= \sqrt{100}$$
- $$= 10 \quad \checkmark$$



B. You've just learned how to verify that points (x, y) are an equal distance from a given point and a given line

Now try Exercises 23 through 26 ▶

C. Characteristics of the Conic Sections

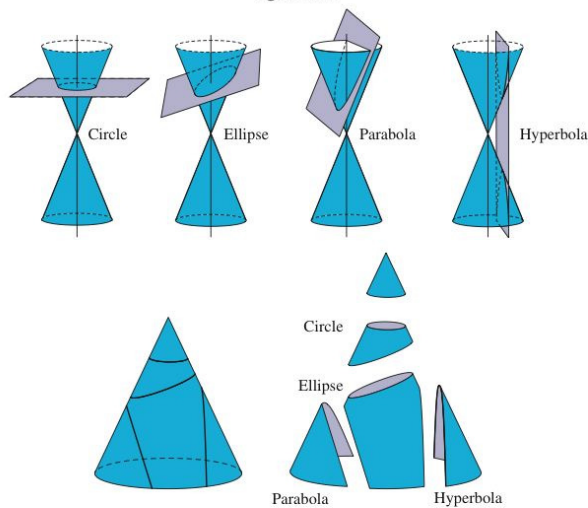


WORTHY OF NOTE

If the plane *does* go through the vertex, the result is a single point, a single line (if the plane contains the generator), or a pair of intersecting lines (if the plane contains the axis).

Examples 1 and 2 bring us one step closer to the wider application of these ideas in a study of the conic sections. But before the connection is clearly made, we'll introduce some background on this family of curves. In common use, a cone might bring to mind the conical paper cups found at a water cooler. The point of the cone is called the **vertex** and the sheet of paper forming the sides is called a **nappe**. In mathematical terms, a cone has two nappes, formed by rotating a nonvertical line (called the generator), about a vertical line (called the axis), at their point of intersection—the vertex (see Figure 10.5). The conic sections are so named because all curves in the family can be formed by a *section* of the cone, or more precisely the intersection of a plane and a cone. Figure 10.6 shows that if the plane does not go through the vertex, the intersection will produce a circle, ellipse, parabola, or hyperbola.

Figure 10.6

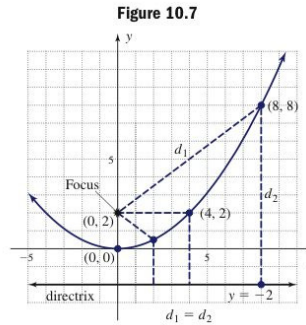


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The connection we seek to make is that each conic section can be defined in terms of the distance between points in the plane, as in Example 1, or the distance between a given point and a line, as in Example 2. In Example 1, we noted the points $(-4, -2)$, $(4, -2)$, and $(4, 4)$ were all on a circle of radius 5 with center $(0, 1)$, in line with the analytic definition of a circle: *A circle is the set of all points that are an equal distance (called the radius) from a given point (called the center).*

In Example 2, you may have noticed that the points seemed to form the right branch of a parabola (see Figure 10.7), and in fact, this example illustrates the analytic definition of a parabola: *A parabola is the set of all points that are an equal distance (called the focus), and a given line (called the directrix).*

The focus and directrix are not actually part of the graph, they are simply used to locate points on the graph. For this reason all foci (plural of focus) will be represented by a "*" symbol rather than a point.



EXAMPLE 3 ▶ Finding an Equation for All Points That Form a Certain Parabola

With Example 2 as a pattern, use the analytic definition to find a formula (equation) for the set of all points that form the parabola.

Solution ▶ Use the ordered pair (x, y) to represent an arbitrary point on the parabola. Since any point on the line $y = -2$ has coordinates $(x, -2)$, we set the distance from $(x, -2)$ to (x, y) equal to the distance from $(0, 2)$ to (x, y) . The result is

$$\begin{aligned} \sqrt{(x-x)^2 + [y-(-2)]^2} &= \sqrt{(x-0)^2 + (y-2)^2} && \text{distances are equal} \\ \sqrt{(y+2)^2} &= \sqrt{x^2 + (y-2)^2} && \text{simplify} \\ (y+2)^2 &= x^2 + (y-2)^2 && \text{power property} \\ y^2 + 4y + 4 &= x^2 + y^2 - 4y + 4 && \text{expand binomials} \\ 8y &= x^2 && \text{simplify} \\ y &= \frac{1}{8}x^2 && \text{result} \end{aligned}$$

All points satisfying these conditions are on the parabola defined by $y = \frac{1}{8}x^2$.

Now try Exercises 27 and 28 ▶

At this point, it seems reasonable to ask what happens when the distance from the focus to (x, y) is *less than* the distance from the directrix to (x, y) . For example, what if the distance is only two-thirds as long? As you might guess, the result is one of the other conic sections, in this case an ellipse. If the distance from the focus to a point (x, y) is *greater than* the distance from the directrix to (x, y) , one branch of a hyperbola is formed. While we will defer a development of their general equations until later in the chapter, the following diagrams serve to illustrate this relationship for the ellipse, and show why we refer to the conic sections as a *family of curves*. In Figure 10.8, the line segment from the focus to each point on the graph (shown in blue), is exactly two-thirds the length of the line segment from the directrix to the same point (shown in red). Note the graph of these points forms the right half of an ellipse. In Figure 10.9, the lines and points forming the first half are moved to the background to more clearly show the remaining points that form the complete graph.

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Figure 10.8

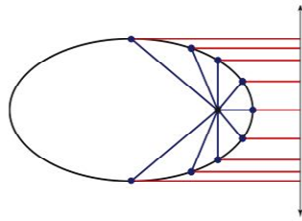
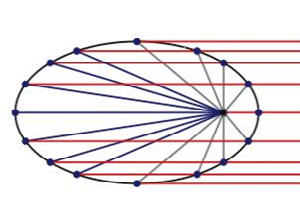


Figure 10.9



EXAMPLE 4 ▶ Finding an Equation for All Points That Form a Certain Ellipse

Suppose we arbitrarily select the point $(1, 0)$ as a focus and the (vertical) line $x = 4$ as the directrix. Use these to find an equation for the set of all points where the distance from the focus to a point (x, y) is $\frac{1}{2}$ the distance from the directrix to (x, y) .

Solution ▶ Since any point on the line $x = 4$ has coordinates $(4, y)$, we have

$$\text{Distance from } (1, 0) \text{ to } (x, y) = \frac{1}{2} [\text{distance from } (4, y) \text{ to } (x, y)] \quad \text{in words}$$

$$\sqrt{(x - 1)^2 + [y - (0)]^2} = \frac{1}{2} \sqrt{(x - 4)^2 + (y - y)^2} \quad \text{resulting equation}$$

$$\sqrt{(x - 1)^2 + y^2} = \frac{1}{2} \sqrt{(x - 4)^2} \quad \text{simplify}$$

$$(x - 1)^2 + y^2 = \frac{1}{4} (x - 4)^2 \quad \text{power property}$$

$$x^2 - 2x + 1 + y^2 = \frac{1}{4} (x^2 - 8x + 16) \quad \text{expand binomials}$$

$$x^2 - 2x + 1 + y^2 = \frac{1}{4} x^2 - 2x + 4 \quad \text{distribute}$$

$$\frac{3}{4} x^2 + y^2 = 3 \quad \text{simplify: } 1x^2 - \frac{1}{4}x^2 = \frac{3}{4}x^2$$

$$3x^2 + 4y^2 = 12 \quad \text{polynomial form}$$

All points satisfying these conditions are on the ellipse defined by $3x^2 + 4y^2 = 12$.

Now try Exercises 29 and 30 ▶

Actually, any given ellipse has two foci (see Figure 10.10) and the equation from Example 4 could also have been developed using the left focus (with the directrix also on the left). This symmetrical relationship leads us to an *alternative definition* for the ellipse, which we will explore further in Section 10.2.

For foci f_1 and f_2 , an ellipse is the set of all points (x, y) where the sum of the distances from f_1 to (x, y) and f_2 to (x, y) is constant.

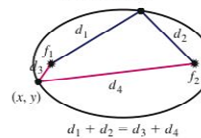
See Figure 10.11 and Exercises 31 and 32.

Both the focus/directrix definition and the two foci definition have merit, and simply tend to call out different characteristics and applications of the ellipse. The hyperbola also has a focus/directrix definition and a two foci definition. See Exercises 33 and 34.

Figure 10.10



Figure 10.11



C. You've just learned how to use the defining characteristics of a conic section to find its equation

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10.1 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

1. Analytical geometry is a study of _____ using the tools of _____.
2. The distance formula is $d =$ _____; the midpoint formula is $M =$ _____.
3. The distance between a point and a line always refers to the _____ distance.
4. The conic sections are formed by the intersection of a _____ and a _____.
5. If a plane intersects a cone at its vertex, the result is a _____, a line, or a pair of _____ lines.
6. A circle is defined relative to an equal distance between two _____. A parabola is defined relative to an equal distance between a _____ and a _____.

► DEVELOPING YOUR SKILLS

The three points given form a right triangle. Find the midpoint of the hypotenuse and verify that the midpoint is an equal distance from all three vertices.

- | | |
|-----------------------|---------------------|
| 7. $P_1 = (-5, 2)$ | 8. $P_1 = (3, 2)$ |
| $P_2 = (1, 2)$ | $P_2 = (3, 14)$ |
| $P_3 = (-5, -6)$ | $P_3 = (8, 2)$ |
| 9. $P_1 = (-2, 1)$ | 10. $P_1 = (0, -5)$ |
| $P_2 = (6, -5)$ | $P_2 = (-6, 4)$ |
| $P_3 = (2, -7)$ | $P_3 = (6, -1)$ |
| 11. $P_1 = (10, -21)$ | 12. $P_1 = (6, -6)$ |
| $P_2 = (-6, -9)$ | $P_2 = (-12, 18)$ |
| $P_3 = (3, 3)$ | $P_3 = (20, 42)$ |

13. Find the equation of the circle that circumscribes the triangle in Exercise 7.
14. Find the equation of the circle that circumscribes the triangle in Exercise 8.
15. Find the equation of the circle that circumscribes the triangle in Exercise 9.

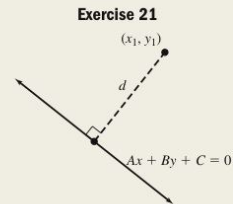
► WORKING WITH FORMULAS

The Perpendicular Distance from a Point to a Line:

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

The perpendicular distance from a point (x_1, y_1) to a given line can be found using the formula shown, where $Ax + By + C = 0$ is the equation of the line in standard form (A, B , and C are integers).

21. Use the formula to verify that $P(-6, 2)$ and $Q(6, 4)$ are an equal distance from the line $y = -\frac{1}{2}x + 3$.



22. Find the value(s) for y that ensure $(1, y)$ is this same distance from $y = -\frac{1}{2}x + 3$.

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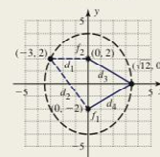
► APPLICATIONS

23. Of the following four points, three are an equal distance from the point $A(0, 1)$ and the line $y = -1$. (a) Identify which three, and (b) find any two additional points that satisfy these conditions.
 $B(-6, 9)$ $C(4, 4)$ $D(-2\sqrt{2}, 6)$ $E(4\sqrt{2}, 8)$
24. Of the following four points, three are an equal distance from the point $P(2, 4)$ and the line $y = -4$. (a) Identify which three, and (b) find any two additional points that satisfy these conditions.
 $Q(-10, 9)$ $R(2 + 4\sqrt{2}, 3)$ $S(10, 4)$
 $T(2 - 4\sqrt{5}, 5)$
25. Consider a fixed point $(0, -4)$ and a fixed line $y = 4$. Verify that the distance from each point to $(0, -4)$, is equal to the distance from the point to the line $y = 4$.
 $A(4, -1)$ $B\left(10, -\frac{25}{4}\right)$ $C(4\sqrt{2}, -2)$
 $D(8\sqrt{5}, -20)$
26. Consider a fixed point $(0, -2)$ and a fixed line $y = 2$. Verify that the distance from each point to $(0, -2)$, is equal to the distance from the point to the line $y = 2$.
 $P(12, -18)$ $Q\left(6, -\frac{9}{2}\right)$ $R(4\sqrt{5}, -10)$
 $S(4\sqrt{6}, -12)$
27. The points from Exercise 25 are on the graph of a parabola. Find the equation of the parabola.
28. The points from Exercise 26 are on the graph of a parabola. Find the equation of the parabola.
29. Using $(0, -2)$ as the focus and the horizontal line $y = -8$ as the directrix, find an equation for the set of all points (x, y) where the distance from the focus to (x, y) is one-half the distance from the directrix to (x, y) .
30. Using $(4, 0)$ as the focus and the vertical line $x = 9$ as the directrix, find an equation for the set of all

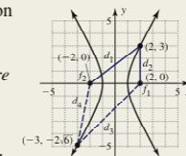
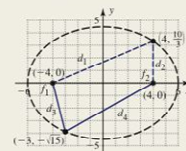
points (x, y) where the distance from the focus to (x, y) is two-thirds the distance from the directrix to (x, y) .

31. From Exercise 29, verify the points $(-3, 2)$ and $(\sqrt{12}, 0)$ are on the ellipse defined by $4x^2 + 3y^2 = 48$. Then verify that $d_1 + d_2 = d_3 + d_4$.
32. From Exercise 30, verify the points $(4, \frac{10}{3})$ and $(-3, -\sqrt{15})$ are on the ellipse defined by $5x^2 + 9y^2 = 180$. Then verify that $d_1 + d_2 = d_3 + d_4$.
33. From the focus/directrix definition of a hyperbola: If the distance from the focus to a point (x, y) is greater than the distance from the directrix to (x, y) , one branch of a hyperbola is formed. Using $(2, 0)$ as the focus and the vertical line $x = \frac{1}{2}$ as the directrix, find an equation for the set of all points (x, y) where the distance from the focus to (x, y) is twice the distance from the directrix to (x, y) .
34. From the two foci definition of a hyperbola: For foci f_1 and f_2 , a hyperbola is the set of all points (x, y) where the difference of the distances from f_1 to (x, y) and f_2 to (x, y) is constant. Verify the points $(2, 3)$ and $(-3, -2\sqrt{6})$ are on the graph of the hyperbola from Exercise 33. Then verify $d_1 - d_2 = d_3 - d_4$.

Exercise 31



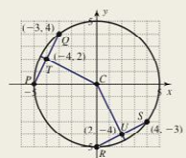
Exercise 32



► EXTENDING THE CONCEPT

35. Do some reading or research on the **orthocenter** of a triangle, and the **centroid** of a triangle. How are they found? What are their properties? Use the ideas and skills from this section to find the (a) orthocenter and (b) centroid of the triangle formed by the points $A(-8, 2)$, $B(-2, -6)$, and $C(4, 0)$.

36. **Properties of a circle:** A theorem from elementary geometry states: If a radius is perpendicular to a chord, it bisects the chord. Verify this is true for the circle, radii, and chords shown.



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10-9

Section 10.2 The Circle and the Ellipse

927

37. Verify that points $C(-2, 3)$ and $D(2\sqrt{2}, \sqrt{6})$ are points on the ellipse with foci at $A(-2, 0)$ and $B(2, 0)$, by verifying $d(AC) + d(BC) = d(AD) + d(BD)$. The expression that results has the form

$\sqrt{A+B} + \sqrt{A-B}$, which prior to the common use of technology had to be simplified using the formula $\sqrt{A+B} + \sqrt{A-B} = \sqrt{a} + \sqrt{b}$, where $a = 2A$ and $b = 4(A^2 - B^2)$. Use this relationship to verify the equation above.

► MAINTAINING YOUR SKILLS

38. (6.4) Verify the following is an identity:

$$\frac{\cos(2x) + \sin^2 x}{1 - \cos^2 x} = \cot^2(x)$$

39. (6.7) Find all solutions in $[0, 2\pi)$

$$-225 = 600 + 825 \sin\left(x + \frac{\pi}{6}\right)$$

40. (4.4) Solve for x in both exact and approximate form:

a. $5 = \frac{10}{1 + 9e^{-0.5x}}$

b. $345 = 5e^{0.4x} + 75$

41. (3.5) Sketch a complete graph of $h(x) = \frac{x^2 - 9}{x^2 - 4}$. Clearly label all intercepts and asymptotes.

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10.2 The Circle and the Ellipse

Learning Objectives

In Section 10.2 you will learn how to:

- A. Use the characteristics of a circle and its graph to understand the equation of an ellipse
- B. Use the equation of an ellipse to graph central and noncentral ellipses
- C. Locate the foci of an ellipse and use the foci and other features to write the equation
- D. Solve applications involving the foci

In Section 10.1, we introduced the equation of an ellipse using analytical geometry and the focus-directrix definition. Here we'll take a different approach, and use the equation of a circle to demonstrate that a circle is simply a special ellipse. In doing so, we'll establish a relationship between the foci and vertices of the ellipse, that enables us to apply these characteristics in context.

A. The Equation and Graph of a Circle

Recall that the equation of a circle with radius r and center at (h, k) is

$$(x - h)^2 + (y - k)^2 = r^2.$$

As in Section 2.1, the standard form can be used to construct the equation of the circle given the center and radius as in Example 1, or to graph the circle as in Example 2.

EXAMPLE 1 ► Determining the Equation of a Circle Given Its Center and Radius

Find the equation of a circle with radius 5 and center at $(2, -1)$.

Solution ► With a center of $(2, -1)$, we have $h = 2$, $k = -1$, and $r = 5$. Making the corresponding substitutions into the standard form we obtain

$$\begin{aligned} (x - h)^2 + (y - k)^2 &= r^2 && \text{standard form} \\ (x - 2)^2 + [y - (-1)]^2 &= 5^2 && \text{substitute 2 for } h, -1 \text{ for } k, \text{ and 5 for } r \\ (x - 2)^2 + (y + 1)^2 &= 25 && \text{simplify} \end{aligned}$$

The equation of this circle is $(x - 2)^2 + (y + 1)^2 = 25$.

Now try Exercises 7 through 12 ►

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If the equation is given in polynomial form, recall that we first complete the square in x and y to identify the center and radius.

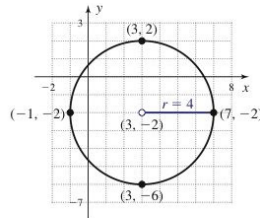
EXAMPLE 2 ▶ **Completing the Square to Graph a Circle**

Find the center and radius of the circle whose equation is given, then sketch its graph: $x^2 + y^2 - 6x + 4y - 3 = 0$.

Solution ▶ Begin by completing the square in both x and y .

$$\begin{aligned} (x^2 - 6x + \underline{\quad}) + (y^2 + 4y + \underline{\quad}) &= 3 && \text{group } x\text{- and } y\text{-terms; add 3} \\ (x^2 - 6x + 9) + (y^2 + 4y + 4) &= 3 + 9 + 4 && \text{complete the square} \\ \text{adds 9 to left side} & \quad \text{adds 4 to left side} & \quad \text{add 9 + 4 to right side} & \\ (x - 3)^2 + (y + 2)^2 &= 16 && \text{factor and simplify} \end{aligned}$$

The center is at $(3, -2)$, with radius is $r = \sqrt{16} = 4$.



Circle
 Center at $(3, -2)$
 Radius: $r = 4$
 Diameter: $2r = 8$
 Endpoints of horizontal diameter
 $(-1, -2)$ and $(7, -2)$
 Endpoints of vertical diameter
 $(3, 2)$ and $(3, -6)$

Now try Exercises 13 through 18 ▶

The equation of a circle in **standard form** provides a useful link to some of the other conic sections, and is obtained by *setting the equation equal to 1*. In the case of a circle, this means we simply divide by r^2 .

$$\begin{aligned} (x - h)^2 + (y - k)^2 &= r^2 && \text{standard form} \\ \frac{(x - h)^2}{r^2} + \frac{(y - k)^2}{r^2} &= 1 && \text{divide by } r^2 \end{aligned}$$

In this form, the value of r in each denominator gives the *horizontal* and *vertical* distances, respectively, from the center to the graph. This is not so important in the case of a circle, since this distance is the same in *any* direction. But for other conics, these horizontal and vertical distances are *not* the same, making the new form a valuable tool for graphing. To distinguish the horizontal from the vertical distance, r^2 is replaced by a^2 in the “ x -term” (horizontal distance), and by b^2 in the “ y -term” (vertical distance).

✓ **A.** You've just learned how to use the characteristics of a circle and its graph to understand the equation of an ellipse

B. The Equation of an Ellipse

It then seems reasonable to ask, “What happens to the graph when $a \neq b$?” To answer, consider the equation from Example 2. We have $\frac{(x - 3)^2}{4^2} + \frac{(y + 2)^2}{4^2} = 1$ (after dividing by 16), which we now compare to $\frac{(x - 3)^2}{4^2} + \frac{(y + 2)^2}{3^2} = 1$, where $a = 4$

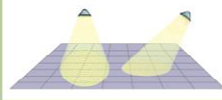
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10-11

Section 10.2 The Circle and the Ellipse

WORTHY OF NOTE

If you point a flashlight at the floor keeping it perpendicular to the ground, a circle is formed with the bulb pointing directly at the center and every point along the outer edge of the beam an equal distance from this center. If you hold the flashlight at an angle, the circle is elongated and becomes an ellipse, with the bulb pointing directly at one focus.



and $b = 3$. The center of the graph is still at $(3, -2)$, since $h = 3$ and $k = -2$ remain unchanged. Substituting $y = -2$ to find additional points, eliminates the y -term and gives two values for x :

$$\begin{aligned} \frac{(x - 3)^2}{4^2} + \frac{(-2 + 2)^2}{3^2} &= 1 && \text{substitute } -2 \text{ for } y \\ \frac{(x - 3)^2}{4^2} + 0 &= 1 && \text{simplify} \\ (x - 3)^2 &= 16 && \text{multiply by } 4^2 = 16 \\ x - 3 &= \pm 4 && \text{property of square roots} \\ x &= 3 \pm 4 && \text{add 3} \\ x &= 7 \text{ and } x = -1 \end{aligned}$$

This shows the horizontal distance from the center to the graph is still $a = 4$, and the points $(-1, -2)$ and $(7, -2)$ are on the graph (see Figure 10.12). Similarly, for $x = 3$ we have $(y + 2)^2 = 9$, giving $y = -5$ and $y = 1$, and showing the vertical distance from the center to the graph is $b = 3$, with points $(3, 1)$ and $(3, -5)$ also on the graph. Using this information to sketch the curve reveals the “circle” is elongated and has become an **ellipse**.

For an ellipse, the longest distance across the graph is called the **major axis**, with the endpoints of the major axis called **vertices**. The segment perpendicular to and bisecting the major axis (with its endpoints on the ellipse) is called the **minor axis**, as shown in see Figure 10.13.

Figure 10.12

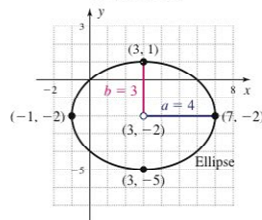
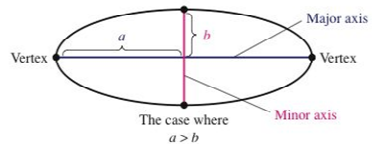


Figure 10.13



- If $a > b$, the major axis is horizontal (parallel to the x -axis) with length $2a$, and the minor axis is vertical with length $2b$ (see Example 3).
- If $b > a$ the major axis is vertical (parallel to the y -axis) with length $2b$, and the minor axis is horizontal with length $2a$ (see Example 4).

Generalizing this observation we obtain the equation of an ellipse in standard form.

The Equation of an Ellipse in Standard Form

Given $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$.

If $a \neq b$ the equation represents the graph of an ellipse with center at (h, k) .

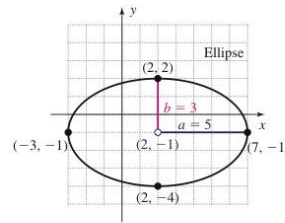
- $|a|$ gives the horizontal distance from center to graph.
- $|b|$ gives the vertical distance from center to graph.

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EXAMPLE 3 ▶ Graphing a Horizontal Ellipse

Sketch the graph of $\frac{(x - 2)^2}{25} + \frac{(y + 1)^2}{9} = 1$.

Solution ▶ Noting $a \neq b$, we have an ellipse with center $(h, k) = (2, -1)$. The horizontal distance from the center to the graph is $a = 5$, and the vertical distance from the center to the graph is $b = 3$. After plotting the corresponding points and connecting them with a smooth curve, we obtain the graph shown.



Now try Exercises 19 through 24 ▶

WORTHY OF NOTE

In general, for the equation $Ax^2 + By^2 = F$ ($A, B, F > 0$), the equation represents a circle if $A = B$, and an ellipse if $A \neq B$.

As with the circle, the equation of an ellipse can be given in polynomial form, and here our knowledge of circles is helpful. For the equation $25x^2 + 4y^2 = 100$, we know the graph cannot be a circle since the coefficients are unequal, and the center of the graph must be at the origin since $h = k = 0$. To actually draw the graph, we convert the equation to standard form.

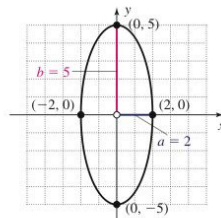
EXAMPLE 4 ▶ Graphing a Vertical Ellipse

For $25x^2 + 4y^2 = 100$, (a) write the equation in standard form and identify the center and the values of a and b , (b) identify the major and minor axes and name the vertices, and (c) sketch the graph.

Solution ▶ The coefficients of x^2 and y^2 are unequal, and 25, 4, and 100 have like signs. The equation represents an ellipse with center at $(0, 0)$. To obtain standard form:

a. $25x^2 + 4y^2 = 100$ given equation
 $\frac{25x^2}{100} + \frac{4y^2}{100} = 1$ divide by 100
 $\frac{x^2}{4} + \frac{y^2}{25} = 1$ standard form
 $\frac{x^2}{2^2} + \frac{y^2}{5^2} = 1$ write denominators in squared form; $a = 2, b = 5$

- b. The result shows $a = 2$ and $b = 5$, indicating the major axis will be vertical and the minor axis will be horizontal. With the center at the origin, the x -intercepts will be $(2, 0)$ and $(-2, 0)$, with the vertices (and y -intercepts) at $(0, 5)$ and $(0, -5)$.
 c. Plotting these intercepts and sketching the ellipse results in the graph shown.



Vertical ellipse
 Center at $(0, 0)$
 Endpoints of major axis (vertices) $(0, -5)$ and $(0, 5)$
 Endpoints of minor axis $(-2, 0)$ and $(2, 0)$
 Length of major axis $2b: 2(5) = 10$
 Length of minor axis $2a: 2(2) = 4$

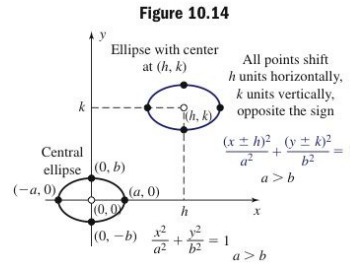
Now try Exercises 25 through 36 ▶

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WORTHY OF NOTE

After writing the equation in standard form, it is possible to end up with a constant that is zero or negative. In the first case, the graph is a single point. In the second case, no graph is possible since roots of the equation will be complex numbers. These are called *degenerate cases*. See **Exercise 78**.

If the center of the ellipse is not at the origin, the polynomial form has additional linear terms and we must first complete the square in x and y , then write the equation in standard form to sketch the graph (see the Reinforcing Basic Concepts feature for more on completing the square). Figure 10.14 illustrates how the central ellipse and the shifted ellipse are related.



EXAMPLE 5 ▶ **Completing the Square to Graph an Ellipse**

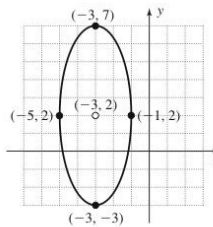
Sketch the graph of $25x^2 + 4y^2 + 150x - 16y + 141 = 0$.

Solution ▶ The coefficients of x^2 and y^2 are unequal and have like signs, and we assume the equation represents an ellipse but wait until we have the factored form to be certain.

$$\begin{aligned}
 25x^2 + 4y^2 + 150x - 16y + 141 &= 0 && \text{given equation (polynomial form)} \\
 25x^2 + 150x + 4y^2 - 16y &= -141 && \text{group like terms; subtract 141} \\
 25(x^2 + 6x + \underline{\quad}) + 4(y^2 - 4y + \underline{\quad}) &= -141 && \text{factor out leading coefficient from each group} \\
 25(x^2 + 6x + 9) + 4(y^2 - 4y + 4) &= -141 + 225 + 16 && \text{complete the square} \\
 \uparrow \text{adds } 25(9) = 225 & \quad \uparrow \text{adds } 4(4) = 16 && \text{add } 225 + 16 \text{ to right} \\
 25(x + 3)^2 + 4(y - 2)^2 &= 100 && \text{factor} \\
 \frac{25(x + 3)^2}{100} + \frac{4(y - 2)^2}{100} &= \frac{100}{100} && \text{divide both sides by 100} \\
 \frac{(x + 3)^2}{4} + \frac{(y - 2)^2}{25} &= 1 && \text{simplify (standard form)} \\
 \frac{(x + 3)^2}{2^2} + \frac{(y - 2)^2}{5^2} &= 1 && \text{write denominators in squared form}
 \end{aligned}$$

The result is a vertical ellipse with center at $(-3, 2)$, with $a = 2$ and $b = 5$. The vertices are a vertical distance of 5 units from center, and the endpoints of the minor axis are a horizontal distance of 2 units from center.

Note this is the same ellipse as in Example 4, but shifted 3 units left and 2 up.



Vertical ellipse
 Center at $(-3, 2)$
 Endpoints of major axis (vertices) $(-3, -3)$ and $(-3, 7)$
 Endpoints of minor axis $(-5, 2)$ and $(-1, 2)$
 Length of major axis $2b: 2(5) = 10$
 Length of minor axis $2a: 2(2) = 4$

Now try Exercises 37 through 44 ▶

B. You've just learned how to use the equation of an ellipse to graph central and noncentral ellipses

C. The Foci of an Ellipse

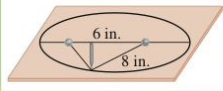
In Section 10.1, we noted that an ellipse could also be defined in terms of two special points called the **foci**. The Museum of Science and Industry in Chicago, Illinois (<http://www.msichicago.org>), has a permanent exhibit called the *Whispering Gallery*. The construction of the room is based on some of the reflective properties of an ellipse. If two people stand at designated points in the room and one of them

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whispers very softly, the other person can hear the whisper quite clearly—even though they are over 40 ft apart! The point where each person stands is a **focus** of the ellipse. This reflective property also applies to light and radiation, giving the ellipse some powerful applications in science, medicine, acoustics, and other areas. To understand and appreciate these applications, we introduce the analytic definition of an ellipse.

WORTHY OF NOTE

You can easily draw an ellipse that satisfies the definition. Press two pushpins (these form the foci of the ellipse) halfway down into a piece of heavy cardboard about 6 in. apart. Take an 8-in. piece of string and loop each end around the pins. Use a pencil to draw the string taut and keep it taut as you move the pencil in a circular motion—and the result is an ellipse! A different length of string or a different distance between the foci will produce a different ellipse.

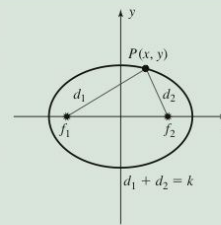


Definition of an Ellipse

Given two fixed points f_1 and f_2 in a plane, an ellipse is the set of all points (x, y) where the distance from f_1 to (x, y) added to the distance from f_2 to (x, y) remains constant.

$$d_1 + d_2 = k$$

The fixed points f_1 and f_2 are called the *foci* of the ellipse, and the points $P(x, y)$ are on the graph of the ellipse.



To find the equation of an ellipse in terms of a and b we combine the definition just given with the distance formula. Consider the ellipse shown in Figure 10.15 (for calculating ease we use a central ellipse). Note the vertices have coordinates $(-a, 0)$ and $(a, 0)$, and the endpoints of the minor axis have coordinates $(0, -b)$ and $(0, b)$ as before. It is customary to assign foci the coordinates $f_1 \rightarrow (-c, 0)$ and $f_2 \rightarrow (c, 0)$. We can calculate the distance between $(c, 0)$ and any point $P(x, y)$ on the ellipse using the distance formula:

$$\sqrt{(x - c)^2 + (y - 0)^2}$$

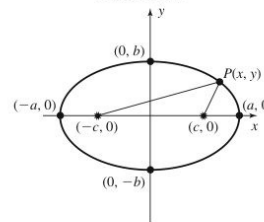
Likewise the distance between $(-c, 0)$ and any point (x, y) is

$$\sqrt{(x + c)^2 + (y - 0)^2}$$

According to the definition, the sum must be constant:

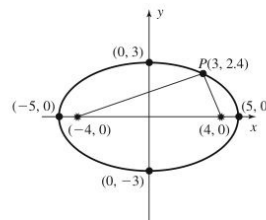
$$\sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = k$$

Figure 10.15



EXAMPLE 6 ▶ Finding the Value of k from the Definition of an Ellipse

Use the definition of an ellipse and the diagram given to determine the constant k used for this ellipse following (also see the following *Worthy of Note*). Note that $a = 5$, $b = 3$, and $c = 4$.



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Solution ▶

WORTHY OF NOTE

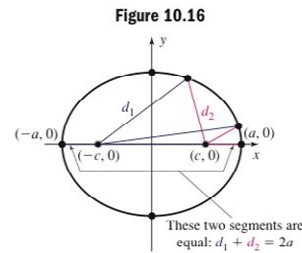
Note that if the foci are coincident (both at the origin) the “ellipse” will actually be a circle with radius $\frac{k}{2}$. $\sqrt{x^2 + y^2} + \sqrt{x^2 + y^2} = k$ leads to $x^2 + y^2 = \frac{k^2}{4}$. In Example 1 we found $k = 10$, giving $\frac{10}{2} = 5$, and if we used the “string” to draw the circle, the pencil would be 5 units from the center, creating a circle of radius 5.

$$\begin{aligned} \sqrt{(x-c)^2 + (y-0)^2} + \sqrt{(x+c)^2 + (y-0)^2} &= k && \text{given} \\ \sqrt{(3-4)^2 + (2.4-0)^2} + \sqrt{(3+4)^2 + (2.4-0)^2} &= k && \text{substitute} \\ \sqrt{(-1)^2 + 2.4^2} + \sqrt{7^2 + 2.4^2} &= k && \text{add} \\ \sqrt{6.76} + \sqrt{54.76} &= k && \text{simplify radicals} \\ 2.6 + 7.4 &= k && \text{compute square roots} \\ 10 &= k && \text{result} \end{aligned}$$

The constant used for this ellipse is 10 units.

Now try Exercises 45 through 48 ▶

In Example 6, the sum of the distances could also be found by moving the point (x, y) to the location of a vertex $(a, 0)$, then using the symmetry of the ellipse. The sum is identical to the length of the major axis, since the overlapping part of the string from $(c, 0)$ to $(a, 0)$ is the same length as from $(-a, 0)$ to $(-c, 0)$ (see Figure 10.16). This shows the constant k is equal to $2a$ regardless of the distance between foci.



As we noted, the result is

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a \quad \text{substitute } 2a \text{ for } k$$

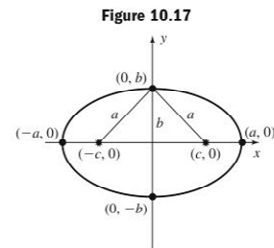
The details for simplifying this expression are given in Appendix III, and the result is very close to the standard form seen previously:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

By comparing the standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$, we might suspect that $b^2 = a^2 - c^2$, and this is indeed the case. Note from Example 6 the relationship yields

$$\begin{aligned} b^2 &= a^2 - c^2 \\ 3^2 &= 5^2 - 4^2 \\ 9 &= 25 - 16 \end{aligned}$$

Additionally, when we consider that $(0, b)$ is a point on the ellipse, the distance from $(0, b)$ to $(c, 0)$ must be equal to a due to symmetry (the “constant distance” used to form the ellipse is always $2a$). We then see in Figure 10.17, that $b^2 + c^2 = a^2$ (Pythagorean Theorem), yielding $b^2 = a^2 - c^2$ as above.



With this development, we now have the ability to locate the foci of any ellipse—an important step toward using the ellipse in practical applications. Because we're often asked to find the location of the foci, it's best to rewrite the relationship in terms of c^2 , using absolute value bars to allow for a major axis that is vertical: $c^2 = |a^2 - b^2|$.

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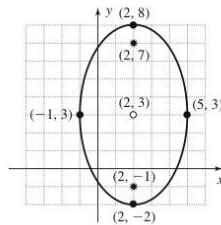
EXAMPLE 7 ▶ **Completing the Square to Graph an Ellipse and Locate the Foci**

For the ellipse defined by $25x^2 + 9y^2 - 100x - 54y - 44 = 0$, find the coordinates of the center, vertices, foci, and endpoints of the minor axis. Then sketch the graph.

Solution ▶

$$\begin{aligned}
 25x^2 + 9y^2 - 100x - 54y - 44 &= 0 && \text{given} \\
 25x^2 - 100x + 9y^2 - 54y &= 44 && \text{group terms; add 44} \\
 25(x^2 - 4x + \underline{\quad}) + 9(y^2 - 6y + \underline{\quad}) &= 44 && \text{factor out lead coefficients} \\
 \begin{array}{c} \uparrow \\ 25(x^2 - 4x + 4) \end{array} + \begin{array}{c} \uparrow \\ 9(y^2 - 6y + 9) \end{array} &= 44 + 100 + 81 && \text{add } 100 + 81 \text{ to right-hand side} \\
 \begin{array}{c} \uparrow \\ \text{adds } 25(4) = 100 \end{array} & \begin{array}{c} \uparrow \\ \text{adds } 9(9) = 81 \end{array} && \\
 25(x - 2)^2 + 9(y - 3)^2 &= 225 && \text{factored form} \\
 \frac{25(x - 2)^2}{225} + \frac{9(y - 3)^2}{225} &= \frac{225}{225} && \text{divide by 225} \\
 \frac{(x - 2)^2}{9} + \frac{(y - 3)^2}{25} &= 1 && \text{simplify (standard form)} \\
 \frac{(x - 2)^2}{3^2} + \frac{(y - 3)^2}{5^2} &= 1 && \text{write denominators in squared form}
 \end{aligned}$$

The result shows a vertical ellipse with $a = 3$ and $b = 5$. The center of the ellipse is at $(2, 3)$. The vertices are a vertical distance of $b = 5$ units from center at $(2, 8)$ and $(2, -2)$. The endpoints of the minor axis are a horizontal distance of $a = 3$ units from center at $(-1, 3)$ and $(5, 3)$. To locate the foci, we use the foci formula for an ellipse: $c^2 = |a^2 - b^2|$, giving $c^2 = |3^2 - 5^2| = 16$. This shows the foci “*” are located a vertical distance of 4 units from center at $(2, 7)$ and $(2, -1)$.



Vertical ellipse
 Center at $(2, 3)$
 Endpoints of major axis (vertices) $(2, 8)$ and $(2, -2)$
 Endpoints of minor axis $(-1, 3)$ and $(5, 3)$
 Location of foci $(2, 7)$ and $(2, -1)$
 Length of major axis: $2b = 2(5) = 10$
 Length of minor axis: $2a = 2(3) = 6$

Now try Exercises 49 through 54 ▶

For future reference, remember the foci of an ellipse always occur on the major axis, with $a > c$ and $a^2 > c^2$ for a horizontal ellipse. This makes it easier to remember the **foci formula** for ellipses: $c^2 = |a^2 - b^2|$. Since a^2 is larger, it must be decreased by b^2 to equal c^2 .

If any two of the values for a , b , and c are known, the relationship between them can be used to construct the equation of the ellipse.

EXAMPLE 8 ▶ **Finding the Equation of an Ellipse**

Find the equation of the ellipse (in standard form) that has foci at $(0, -2)$ and $(0, 2)$, with a minor axis 6 units in length.

Solution ▶

Since the foci must be on the major axis, we know this is a vertical and central ellipse with $c = 2$ and $c^2 = 4$. The minor axis has a length of $2a = 6$ units, meaning $a = 3$ and $a^2 = 9$. To find b^2 , use the foci equation and solve.

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10-17

Section 10.2 The Circle and the Ellipse

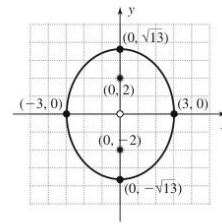
LOOKING AHEAD

For the hyperbola, we'll find that $c > a$, and the formula for the foci of a hyperbola will be $c^2 = a^2 + b^2$.

C. You've just learned how to locate the foci of an ellipse and use the foci and other features to write the equation

$$\begin{aligned}
 c^2 &= |a^2 - b^2| && \text{foci equation (ellipse)} \\
 4 &= |9 - b^2| && \text{substitute} \\
 -4 &= 9 - b^2 && 4 = 9 - b^2 \text{ solve} \\
 b^2 &= 13 && b^2 = 5 \text{ result}
 \end{aligned}$$

Since we know b^2 must be greater than a^2 (the major axis is always longer), $b^2 = 5$ can be discarded. The standard form is $\frac{x^2}{3^2} + \frac{y^2}{(\sqrt{13})^2} = 1$.



Now try Exercises 55 through 64 ▶

D. Applications Involving Foci

Applications involving the foci of a conic section can take various forms. In many cases, only partial information about the conic section is available and the ideas from Example 8 must be used to “fill in the gaps.” In other applications, we must rewrite a known or given equation to find information related to the values of a , b , and c .

EXAMPLE 9 ▶ Solving Applications Using the Characteristics of an Ellipse

In Washington, D.C., there is a park called the *Ellipse* located between the White House and the Washington Monument. The park is surrounded by a path that forms an ellipse with the length of the major axis being about 1502 ft and the minor axis having a length of 1280 ft. Suppose the park manager wants to install water fountains at the location of the foci. Find the distance between the fountains rounded to the nearest foot.

Solution ▶ Since the major axis has length $2a = 1502$, we know $a = 751$ and $a^2 = 564,001$. The minor axis has length $2b = 1280$, meaning $b = 640$ and $b^2 = 409,600$. To find c , use the foci equation:

$$\begin{aligned}
 c^2 &= a^2 - b^2 \\
 &= 564,001 - 409,600 \\
 &= 154,401 \\
 c &\approx -393 \text{ and } c \approx 393
 \end{aligned}$$

The distance between the water fountains would be $2(393) = 786$ ft.

D. You've just learned how to solve applications involving the foci

Now try Exercises 65 through 76 ▶

10.2 EXERCISES

▶ **CONCEPTS AND VOCABULARY**

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

- For an ellipse, the relationship between a , b , and c is given by the foci equation _____, since $c < a$ or $c < b$.
- The greatest distance across an ellipse is called the _____ and the endpoints are called _____.

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3. For a vertical ellipse, the length of the minor axis is _____ and the length of the major axis is _____.
5. Explain/Discuss how the relations $a > b$, $a = b$ and $a < b$ affect the graph of a conic section with equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.
4. To write the equation $2x^2 + y^2 - 6x = 7$ in standard form, _____ the _____ in x .
6. Suppose foci are located at $(-3, 2)$ and $(5, 2)$. Discuss/Explain the conditions necessary for the graph to be an ellipse.

► DEVELOPING YOUR SKILLS

Find the equation of a circle satisfying the conditions given.

7. center $(0, 0)$, radius 7
8. center $(0, 0)$, radius 9
9. center $(5, 0)$, radius $\sqrt{3}$
10. center $(0, 4)$, radius $\sqrt{5}$
11. diameter has endpoints $(4, 9)$ and $(-2, 1)$
12. diameter has endpoints $(-2, -3)$, and $(3, 9)$

Identify the center and radius of each circle, then sketch its graph.

13. $x^2 + y^2 - 12x - 10y + 52 = 0$
14. $x^2 + y^2 + 8x - 6y - 11 = 0$
15. $x^2 + y^2 - 4x + 10y + 4 = 0$
16. $x^2 + y^2 + 4x + 6y - 3 = 0$
17. $x^2 + y^2 + 6x - 5 = 0$
18. $x^2 + y^2 - 8y - 5 = 0$

Sketch the graph of each ellipse.

19. $\frac{(x-1)^2}{9} + \frac{(y-2)^2}{16} = 1$
20. $\frac{(x-3)^2}{4} + \frac{(y-1)^2}{25} = 1$
21. $\frac{(x-2)^2}{25} + \frac{(y+3)^2}{4} = 1$
22. $\frac{(x+5)^2}{1} + \frac{(y-2)^2}{16} = 1$
23. $\frac{(x+1)^2}{16} + \frac{(y+2)^2}{9} = 1$
24. $\frac{(x+1)^2}{36} + \frac{(y+3)^2}{9} = 1$

For each exercise, (a) write the equation in standard form, then identify the center and the values of a and b , (b) state the coordinates of the vertices and the coordinates of the endpoints of the minor axis, and (c) sketch the graph.

25. $x^2 + 4y^2 = 16$
26. $9x^2 + y^2 = 36$
27. $16x^2 + 9y^2 = 144$
28. $25x^2 + 9y^2 = 225$
29. $2x^2 + 5y^2 = 10$
30. $3x^2 + 7y^2 = 21$

Identify each equation as that of an ellipse or circle, then sketch its graph.

31. $(x+1)^2 + 4(y-2)^2 = 16$
32. $9(x-2)^2 + (y+3)^2 = 36$
33. $2(x-2)^2 + 2(y+4)^2 = 18$
34. $(x-6)^2 + y^2 = 49$
35. $4(x-1)^2 + 9(y-4)^2 = 36$
36. $25(x-3)^2 + 4(y+2)^2 = 100$

Complete the square in both x and y to write each equation in standard form. Then draw a complete graph of the relation and identify all important features.

37. $4x^2 + y^2 + 6y + 5 = 0$
38. $x^2 + 3y^2 + 8x + 7 = 0$
39. $x^2 + 4y^2 - 8y + 4x - 8 = 0$
40. $3x^2 + y^2 - 8y + 12x - 8 = 0$
41. $5x^2 + 2y^2 + 20y - 30x + 75 = 0$
42. $4x^2 + 9y^2 - 16x + 18y - 11 = 0$
43. $2x^2 + 5y^2 - 12x + 20y - 12 = 0$
44. $6x^2 + 3y^2 - 24x + 18y - 3 = 0$

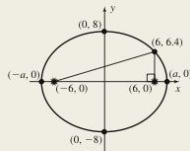
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10-19

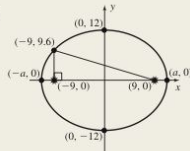
Section 10.2 The Circle and the Ellipse

Use the definition of an ellipse to find the constant k used for each ellipse (figures are not drawn to scale).

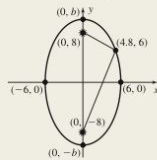
45.



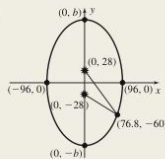
46.



47.



48.



Find the coordinates of the (a) center, (b) vertices, (c) foci, and (d) endpoints of the minor axis. Then (e) sketch the graph.

- 49. $4x^2 + 25y^2 - 16x - 50y - 59 = 0$
- 50. $9x^2 + 16y^2 - 54x - 64y + 1 = 0$
- 51. $25x^2 + 16y^2 - 200x + 96y + 144 = 0$
- 52. $49x^2 + 4y^2 + 196x - 40y + 100 = 0$

► WORKING WITH FORMULAS

63. Area of an Ellipse: $A = \pi ab$
 The area of an ellipse is given by the formula shown, where a is the distance from the center to the graph in the horizontal direction and b is the distance from center to graph in the vertical direction. Find the area of the ellipse defined by $16x^2 + 9y^2 = 144$.

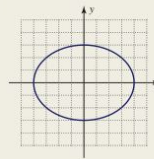
- 53. $6x^2 + 24x + 9y^2 + 36y + 6 = 0$
- 54. $5x^2 - 50x + 2y^2 - 12y + 93 = 0$

Find the equation of an ellipse (in standard form) that satisfies the following conditions:

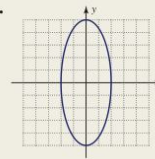
- 55. vertices at $(-6, 0)$ and $(6, 0)$; foci at $(-4, 0)$ and $(4, 0)$
- 56. vertices at $(-8, 0)$ and $(8, 0)$; foci at $(-5, 0)$ and $(5, 0)$
- 57. foci at $(3, -6)$ and $(3, 2)$; length of minor axis: 6 units
- 58. foci at $(-4, -3)$ and $(8, -3)$; length of minor axis: 8 units

Use the characteristics of an ellipse and the graph given to write the related equation and find the location of the foci.

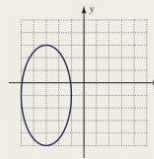
59.



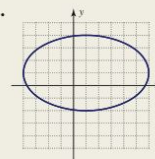
60.



61.



62.

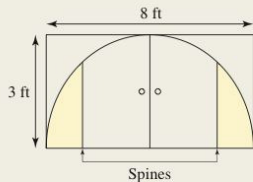


64. The Perimeter of an Ellipse: $P = 2\pi \sqrt{\frac{a^2 + b^2}{2}}$
 The perimeter of an ellipse can be approximated by the formula shown, where a represents the length of the semimajor axis and b represents the length of the semiminor axis. Find the perimeter of the ellipse defined by the equation $\frac{x^2}{49} + \frac{y^2}{4} = 1$.

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► APPLICATIONS

65. Decorative fireplaces: A bricklayer intends to build an elliptical fireplace 3 ft high and 8 ft wide, with two glass doors that open at the middle. The hinges to these doors are to be screwed onto a spine that is perpendicular to the hearth and goes through the foci of the ellipse. How far from center will the spines be located? How tall will each spine be?

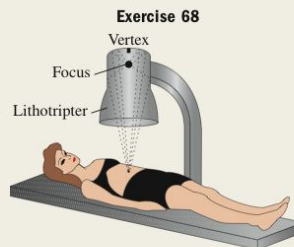


66. Decorative gardens: A retired math teacher decides to present her husband with a beautiful elliptical garden to help celebrate their 50th anniversary. The ellipse is to be 8 m long and 5 m across, with decorative fountains located at the foci. How far from the center of the ellipse should the fountains be located (round to the nearest 100th of a meter)? How far apart are the fountains?

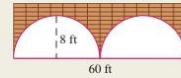
67. Attracting attention to art: As part of an art show, a gallery owner asks a student from the local university to design a unique exhibit that will highlight one of the more significant pieces in the collection, an ancient sculpture. The student decides to create an elliptical showroom with reflective walls, with a rotating laser light on a stand at one focus, and the sculpture placed at the other focus on a stand of equal height. The laser light then points continually at the sculpture as it rotates. If the elliptical room is 24 ft long and 16 ft wide, how far from the center of the ellipse should the stands be located (round to the nearest 10th of a foot)? How far apart are the stands?

68. Medical procedures: The medical procedure called *lithotripsy* is a noninvasive medical procedure that is used to break up kidney and bladder stones in the body. A machine called a *lithotripter* uses its three-dimensional semielliptical shape and the foci properties of an ellipse to concentrate shock waves generated at one focus, on a kidney stone located at the other focus (see diagram — not drawn to scale). If the lithotripter has a length (semimajor axis) of 16 cm and a radius (semiminor axis) of 10 cm, how far from the vertex should a kidney stone be located

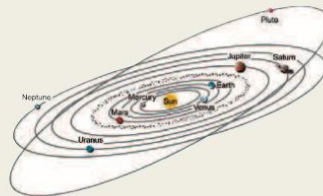
for the best result? Round to the nearest hundredth.



69. Elliptical arches: In some situations, bridges are built using uniform elliptical archways as shown in the figure given. Find the equation of the ellipse forming each arch if it has a total width of 30 ft and a maximum center height (above level ground) of 8 ft. What is the height of a point 9 ft to the right of the center of each arch?



70. Elliptical arches: An elliptical arch bridge is built across a one lane highway. The arch is 20 ft across and has a maximum center height of 12 ft. Will a farm truck hauling a load 10 ft wide with a clearance height of 11 ft be able to go through the bridge without damage? (*Hint:* See Exercise 69.)

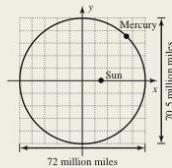


As a planet orbits around the Sun, it traces out an ellipse. If the center of the ellipse were placed at (0, 0) on a coordinate grid, the Sun would be actually off-centered (located at the focus of the ellipse). Use this information and the graphs provided to complete Exercises 71 through 74.

71. Orbit of Mercury: The approximate orbit of the planet Mercury is shown in the figure given. Find an equation that models this orbit.

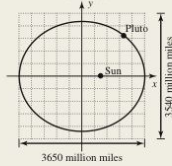
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Exercise 71



72. Orbit of Pluto: The approximate orbit of the dwarf planet Pluto is shown in the figure given. Find an equation that models this orbit.

Exercise 72



73. Planetary orbits: Except for small variations, a planet's orbit around the Sun is elliptical with the Sun at one focus. The aphelion (maximum distance from the Sun) of the planet Mars is approximately 156 million miles, while the perihelion (minimum distance from the Sun) of Mars is about 128 million miles. Use this information to find the lengths of the semimajor and semiminor axes, rounded to the nearest million. If Mars has an orbital velocity of 54,000 miles per hour

(1.296 million miles per day), how many days does it take Mars to orbit the Sun? (*Hint:* Use the formula from Exercise 64).

74. Planetary orbits: The aphelion (maximum distance from the Sun) of the planet Saturn is approximately 940 million miles, while the perihelion (minimum distance from the Sun) of Saturn is about 840 million miles. Use this information to find the lengths of the semimajor and semiminor axes, rounded to the nearest million. If Saturn has an orbital velocity of 21,650 miles per hour (about 0.52 million miles per day), how many days does it take Saturn to orbit the Sun? How many years?

75. Area of a race track: Suppose the *Toronado 500* is a car race that is run on an elliptical track. The track is bounded by two ellipses with equations of $4x^2 + 9y^2 = 900$ and $9x^2 + 25y^2 = 900$, where x and y are in hundreds of yards. Use the formula given in Exercise 63 to find the area of the race track.

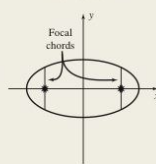
76. Area of a border: The table cloth for a large oval table is elliptical in shape. It is designed with two concentric ellipses (one within the other) as shown in the figure. The equation of the outer ellipse is $9x^2 + 25y^2 = 225$, and the equation of the inner ellipse is $4x^2 + 16y^2 = 64$ with x and y in feet. Use the formula given in Exercise 63 to find the area of the border of the tablecloth.



► **EXTENDING THE THOUGHT**

77. When graphing the conic sections, it is often helpful to use what is called a **focal chord**, as it gives additional points on the graph with very little effort. A focal chord is a line segment through a focus (perpendicular to the major or transverse axis), with the endpoints on the graph. For an ellipse, the length of the focal chord is given by $L = \frac{2m^2}{n}$, where m is the length of the semiminor axis, and n is the length of the semimajor axis. The focus will always be the midpoint of this line segment. Find the length of the focal chord for the ellipse

Exercise 77



$\frac{x^2}{81} + \frac{y^2}{36} = 1$ and the coordinates of the endpoints. Verify (by substituting into the equation) that these endpoints are indeed points on the graph, then use them to help complete the graph.

78. For the equation $6x^2 + 36x + 3y^2 - 24y + 74 = -28$, does the equation appear to be that of a circle, ellipse, or parabola? Write the equation in factored form. What do you notice? What can you say about the graph of this equation?

79. Verify that for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the length of the focal chord is $\frac{2b^2}{a}$.

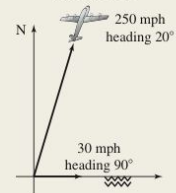
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► MAINTAINING YOUR SKILLS

80. (4.4) Evaluate the expression using the change-of-base formula: $\log_5 20$.
81. (3.8) The resistance R to current flow in an electrical wire varies directly as the length L of the wire and inversely as the square of its diameter d . (a) Write the equation of variation; (b) find the constant of variation if a wire 2 m long with diameter $d = 0.005$ m has a resistance of 240 ohms (Ω); and (c) find the resistance in a similar wire 3 m long and 0.006 m in diameter.

82. (7.6) Use De Moivre's theorem to compute the value of $z = (1 - \sqrt{3}i)^6$.
83. (8.4) Find the true direction and groundspeed of the airplane shown, given the direction and speed of the wind (indicated in blue).

Exercise 83



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10.3 The Hyperbola

Learning Objectives

In Section 10.3 you will learn how to:

- A.** Use the equation of a hyperbola to graph central and noncentral hyperbolas
- B.** Distinguish between the equations of a circle, ellipse, and hyperbola
- C.** Locate the foci of a hyperbola and use the foci and other features to write its equation
- D.** Solve applications involving foci

As seen in Section 10.1 (see Figure 10.18), a hyperbola is a conic section formed by a plane that cuts both nappes of a right circular cone. A hyperbola has two symmetric parts called **branches**, which open in opposite directions. Although the branches appear to resemble parabolas, we will soon discover they are actually a very different curve.

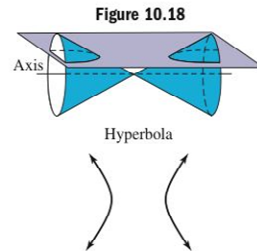


Figure 10.18

A. The Equation of a Hyperbola

In Section 10.2, we noted that for the equation $Ax^2 + By^2 = F$, if $A = B$, the equation is that of a circle, if $A \neq B$, the equation represents an ellipse. Both cases contain a *sum* of second-degree terms. Perhaps driven by curiosity, we might wonder what happens if the equation has a *difference* of second-degree terms. Consider the equation $9x^2 - 16y^2 = 144$. It appears the graph will be centered at $(0, 0)$ since no shifts are applied (h and k are both zero). Using the intercept method to graph this equation reveals an entirely new curve, called a *hyperbola*.

EXAMPLE 1 ▶ Graphing a Central Hyperbola

Graph the equation $9x^2 - 16y^2 = 144$ using intercepts and additional points as needed.

Solution ▶

$$\begin{aligned}
 9x^2 - 16y^2 &= 144 && \text{given} \\
 9(0)^2 - 16y^2 &= 144 && \text{substitute 0 for } x \\
 -16y^2 &= 144 && \text{simplify} \\
 y^2 &= -9 && \text{divide by } -16
 \end{aligned}$$

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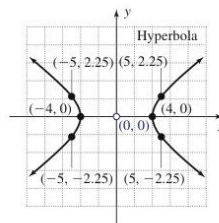
Since y^2 can never be negative, we conclude that the graph has *no y-intercepts*. Substituting $y = 0$ to find the *x-intercepts* gives

$$\begin{aligned} 9x^2 - 16y^2 &= 144 && \text{given} \\ 9x^2 - 16(0)^2 &= 144 && \text{substitute 0 for } y \\ 9x^2 &= 144 && \text{simplify} \\ x^2 &= 16 && \text{divide by 9} \\ x = \sqrt{16} &\text{ and } x = -\sqrt{16} && \text{square root property} \\ x = 4 &\text{ and } x = -4 && \text{simplify} \\ (4, 0) &\text{ and } (-4, 0) && \text{x-intercepts} \end{aligned}$$

Knowing the graph has no *y-intercepts*, we select inputs greater than 4 and less than -4 to help sketch the graph. Using $x = 5$ and $x = -5$ yields

$9x^2 - 16y^2 = 144$	given	$9x^2 - 16y^2 = 144$
$9(5)^2 - 16y^2 = 144$	substitute for x	$9(-5)^2 - 16y^2 = 144$
$9(25) - 16y^2 = 144$	$5^2 = (-5)^2 = 25$	$9(25) - 16y^2 = 144$
$225 - 16y^2 = 144$	simplify	$225 - 16y^2 = 144$
$-16y^2 = -81$	subtract 225	$-16y^2 = -81$
$y^2 = \frac{81}{16}$	divide by -16	$y^2 = \frac{81}{16}$
$y = \frac{9}{4}$ $y = -\frac{9}{4}$	square root property	$y = \frac{9}{4}$ $y = -\frac{9}{4}$
$y = 2.25$ $y = -2.25$	decimal form	$y = 2.25$ $y = -2.25$
$(5, 2.25)$ $(5, -2.25)$	ordered pairs	$(-5, 2.25)$ $(-5, -2.25)$

Plotting these points and connecting them with a smooth curve, while *knowing there are no y-intercepts*, produces the graph in the figure. The point at the origin (in blue) is not a part of the graph, and is given only to indicate the "center" of the hyperbola.



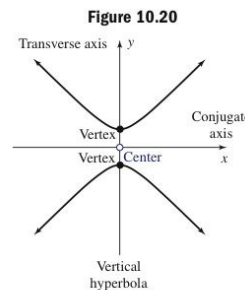
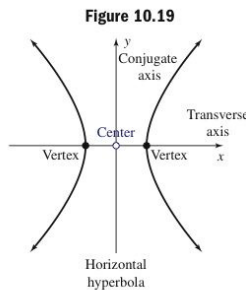
Now try Exercises 7 through 22 ►

Since the hyperbola crosses a horizontal line of symmetry, it is referred to as a **horizontal hyperbola**. The points $(-4, 0)$ and $(4, 0)$ are called **vertices**, and the **center** of the hyperbola is always the point halfway between them. If the center is at the origin, we have a **central hyperbola**. The line passing through the center and both vertices is called the **transverse axis** (vertices are always on the transverse axis), and the line passing through the center and perpendicular to this axis is called the **conjugate axis** (see Figure 10.19).

In Example 1, the coefficient of x^2 was positive and we were subtracting $16y^2$: $9x^2 - 16y^2 = 144$. The result was a horizontal hyperbola. If the y^2 -term is

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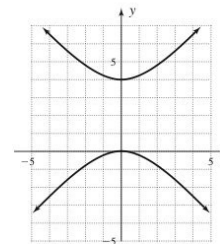
positive and we subtract the term containing x^2 , the result is a **vertical hyperbola** (Figure 10.20).



EXAMPLE 2 ▶ Identifying the Axes, Vertices, and Center of a Hyperbola from Its Graph

For the hyperbola shown, state the location of the vertices and the equation of the transverse axis. Then identify the location of the center and the equation of the conjugate axis.

Solution ▶ By inspection we locate the vertices at (0, 0) and (0, 4). The equation of the transverse axis is $x = 0$. The center is halfway between the vertices at (0, 2), meaning the equation of the conjugate axis is $y = 2$.



Now try Exercises 23 through 26 ▶

Standard Form

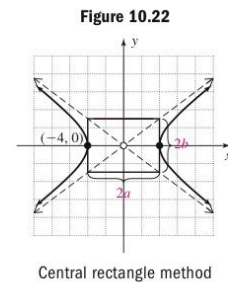
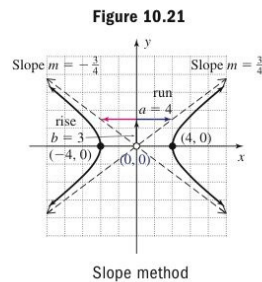
As with the ellipse, the polynomial form of the equation is helpful for *identifying* hyperbolas, but not very helpful when it comes to *graphing* a hyperbola (since we still must go through the laborious process of finding additional points). For graphing, standard form is once again preferred. Consider the hyperbola $9x^2 - 16y^2 = 144$ from Example 1. To write the equation in standard form, we divide by 144 and obtain $\frac{x^2}{4^2} - \frac{y^2}{3^2} = 1$. By comparing the standard form to the graph,

we note $a = 4$ represents the distance from center to vertices, similar to the way we used a previously. But since the graph has no y -intercepts, what could $b = 3$ represent? The answer lies in the fact that branches of a hyperbola are **asymptotic**, meaning they will approach and become very close to imaginary lines that can be used to sketch the graph.

For a central hyperbola, the slopes of the asymptotic lines are given by the ratios $\frac{b}{a}$ and $-\frac{b}{a}$, with the related equations being $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$. The graph from Example 1 is repeated in Figure 10.21, with the asymptotes drawn. For a clearer understanding of how the equations for the asymptotes were determined, see **Exercise 88**.

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A second method of drawing the asymptotes involves drawing a **central rectangle** with dimensions $2a$ by $2b$, as shown in Figure 10.22. The asymptotes will be the *extended diagonals* of this rectangle. This brings us to the equation of a hyperbola in standard form.



The Equation of a Hyperbola in Standard Form

The equation

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

represents a *horizontal* hyperbola with center (h, k)

- *transverse axis* $y = k$
- *conjugate axis* $x = h$
- $|a|$ gives the distance from center to vertices.

The equation

$$\frac{(y - k)^2}{b^2} - \frac{(x - h)^2}{a^2} = 1$$

represents a *vertical* hyperbola with center (h, k)

- *transverse axis* $x = h$
- *conjugate axis* $y = k$
- $|b|$ gives the distance from center to vertices.

- Asymptotes can be drawn by starting at (h, k) and using slopes $m = \pm \frac{b}{a}$.

EXAMPLE 3 ▶ Graphing a Hyperbola Using Its Equation in Standard Form

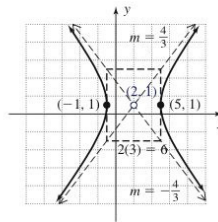
Sketch the graph of $16(x - 2)^2 - 9(y - 1)^2 = 144$. Label the center, vertices, and asymptotes.

Solution ▶ Begin by noting a difference of the second-degree terms, with the x^2 -term occurring first. This means we'll be graphing a horizontal hyperbola whose center is at $(2, 1)$. Continue by writing the equation in standard form.

$$\begin{aligned}
 16(x - 2)^2 - 9(y - 1)^2 &= 144 && \text{given equation} \\
 \frac{16(x - 2)^2}{144} - \frac{9(y - 1)^2}{144} &= \frac{144}{144} && \text{divide by 144} \\
 \frac{(x - 2)^2}{9} - \frac{(y - 1)^2}{16} &= 1 && \text{simplify} \\
 \frac{(x - 2)^2}{3^2} - \frac{(y - 1)^2}{4^2} &= 1 && \text{write denominators in squared form}
 \end{aligned}$$

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Since $a = 3$ the vertices are a horizontal distance of 3 units from the center $(2, 1)$, giving $(2 + 3, 1) \rightarrow (5, 1)$ and $(2 - 3, 1) \rightarrow (-1, 1)$. After plotting the center and vertices, we can begin at the center and count off slopes of $m = \pm \frac{b}{a} = \pm \frac{4}{3}$, or draw a rectangle centered at $(2, 1)$ with dimensions $2(3) = 6$ (horizontal dimension) by $2(4) = 8$ (vertical dimension) to sketch the asymptotes. The complete graph is shown here.



Horizontal hyperbola

- Center at $(2, 1)$
- Vertices at $(-1, 1)$ and $(5, 1)$
- Transverse axis: $y = 1$
- Conjugate axis: $x = 2$
- Width of rectangle (horizontal dimension and distance between vertices)
 $2a = 2(3) = 6$
- Length of rectangle (vertical dimension)
 $2b = 2(4) = 8$

Now try Exercises 27 through 38 ▶

Polynomial Form

If the equation is given as a polynomial in expanded form, complete the square in x and y , then write the equation in standard form.

EXAMPLE 4 ▶ Graphing a Hyperbola by Completing the Square

Graph the equation $9y^2 - x^2 + 54y + 4x + 68 = 0$.

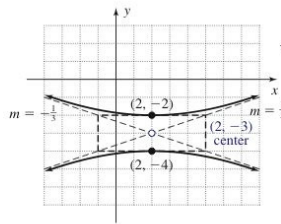
Solution ▶ Since the y^2 -term occurs first, we assume the equation represents a vertical hyperbola, but wait for the factored form to be sure (see **Exercise 87**).

$$\begin{aligned}
 9y^2 - x^2 + 54y + 4x + 68 &= 0 && \text{given} \\
 9y^2 + 54y - x^2 + 4x &= -68 && \text{collect like-variable terms; subtract 68} \\
 9(y^2 + 6y + \underline{\quad}) - 1(x^2 - 4x + \underline{\quad}) &= -68 && \text{factor out 9 from } y\text{-terms and } -1 \text{ from } x\text{-terms} \\
 \underbrace{9(y^2 + 6y + 9)}_{\text{adds } 9(9) = 81} - \underbrace{1(x^2 - 4x + 4)}_{\text{adds } -1(4) = -4} &= -68 + 81 + (-4) && \text{complete the square} \\
 &= -68 + 81 + (-4) && \text{add } 81 + (-4) \text{ to right} \\
 9(y + 3)^2 - 1(x - 2)^2 &= 9 && \text{factor } \rightarrow \text{vertical hyperbola} \\
 \frac{(y + 3)^2}{1} - \frac{(x - 2)^2}{9} &= 1 && \text{divide by 9 (standard form)} \\
 \frac{(y + 3)^2}{1^2} - \frac{(x - 2)^2}{3^2} &= 1 && \text{write denominators in squared form}
 \end{aligned}$$

The center of the hyperbola is $(2, -3)$ with $a = 3$, $b = 1$, and a transverse axis of $x = 2$. The vertices are at $(2, -3 + 1)$ and $(2, -3 - 1) \rightarrow (2, -2)$ and $(2, -4)$. After plotting the center and vertices, we draw a rectangle centered at $(2, -3)$ with a horizontal "width" of $2(3) = 6$ and a vertical "length" of $2(1) = 2$ to sketch the asymptotes. The completed graph is given in the figure.

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A. You've just learned how to use the equation of a hyperbola to graph central and noncentral hyperbolas



Vertical hyperbola
 Center at $(2, -3)$
 Vertices at $(2, -2)$ and $(2, -4)$
 Transverse axis: $x = 2$
 Conjugate axis: $y = -3$
 Width of rectangle (horizontal dimension)
 $2a = 2(3) = 6$
 Length of rectangle (vertical dimension and distance between vertices)
 $2b = 2(1) = 2$

Now try Exercises 39 through 48 ►

B. Distinguishing between the Equations of a Circle, Ellipse, and Hyperbola

So far we've explored numerous graphs of circles, ellipses, and hyperbolas. In Example 5 we'll attempt to identify a given conic section from its equation alone (without graphing the equation). As you've seen, the corresponding equations have unique characteristics that can help distinguish one from the other.

EXAMPLE 5 ► Identifying a Conic Section from Its Equation

Identify each equation as that of a circle, ellipse, or hyperbola. Justify your choice and name the center, but do not draw the graphs.

- a. $y^2 = 36 + 9x^2$ b. $4x^2 = 16 - 4y^2$
 c. $x^2 = 225 - 25y^2$ d. $25x^2 = 100 + 4y^2$
 e. $3(x - 2)^2 + 4(y + 3)^2 = 12$ f. $4(x + 5)^2 = 36 + 9(y - 4)^2$

- Solution ►**
- Writing the equation in factored form gives $y^2 - 9x^2 = 36$ ($h = 0, k = 0$). Since the equation contains a difference of second-degree terms, it is the equation of a (vertical) hyperbola. The center is at $(0, 0)$.
 - Rewriting the equation as $4x^2 + 4y^2 = 16$ and dividing by 4 gives $x^2 + y^2 = 4$. The equation represents a circle of radius 2, with the center at $(0, 0)$.
 - Writing the equation as $x^2 + 25y^2 = 225$ we note a sum of second-degree terms with unequal coefficients. The equation is that of an ellipse, with the center at $(0, 0)$.
 - Rewriting the equation as $25x^2 - 4y^2 = 100$ we note the equation contains a difference of second-degree terms. The equation represents a central (horizontal) hyperbola, whose center is at $(0, 0)$.
 - The equation is in factored form and contains a sum of second-degree terms with unequal coefficients. This is the equation of an ellipse with the center at $(2, -3)$.
 - Rewriting the equation as $4(x + 5)^2 - 9(y - 4)^2 = 36$ we note a difference of second-degree terms. The equation represents a horizontal hyperbola with center $(-5, 4)$.

B. You've just learned how to distinguish between the equations of a circle, ellipse, and hyperbola

Now try Exercises 49 through 60 ►

C. The Foci of a Hyperbola

Like the ellipse, the foci of a hyperbola play an important part in their application. A long distance radio navigation system (called LORAN for short), can be used to determine the location of ships and airplanes and is based on the characteristics of a hyperbola

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(see Exercises 85 and 86). Hyperbolic mirrors are also used in some telescopes, and have the property that a beam of light directed at one focus will be reflected to the second focus. To understand and appreciate these applications, we use the analytic definition of a hyperbola:

Definition of a Hyperbola

Given two fixed points f_1 and f_2 in a plane, a hyperbola is the set of all points (x, y) such that the distance from f_2 to (x, y) subtracted from the distance from f_1 to (x, y) is a positive constant. In symbols,

$$d_1 - d_2 = k, k > 0$$

The fixed points f_1 and f_2 are called the foci of the hyperbola, and the points (x, y) are on the graph of the hyperbola.

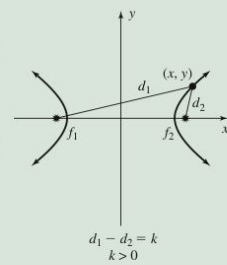
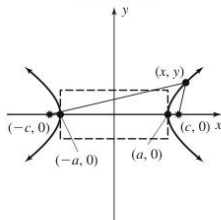


Figure 10.23



As with the analytic definition of the ellipse, it can be shown that the constant k is again equal to $2a$ (for horizontal hyperbolas). To find the equation of a hyperbola in terms of a and b , we use an approach similar to that of the ellipse (see Appendix III), and the result is identical to that seen earlier: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $b^2 = c^2 - a^2$ (see Figure 10.23).

We now have the ability to find the foci of any hyperbola—and can use this information in many significant applications. Since the location of the foci play such an important role, it is best to remember the relationship as $c^2 = a^2 + b^2$ (called the **foci formula** for hyperbolas), noting that for a hyperbola, $c > a$ and $c^2 > a^2$ (also $c > b$ and $c^2 > b^2$).

EXAMPLE 6 ▶ Graphing a Hyperbola and Identifying Its Foci by Completing the Square.

For the hyperbola defined by $7x^2 - 9y^2 - 14x + 72y - 200 = 0$, find the coordinates of the center, vertices, foci, and the dimensions of the central rectangle. Then sketch the graph.

Solution ▶

$$\begin{aligned}
 7x^2 - 9y^2 - 14x + 72y - 200 &= 0 && \text{given} \\
 7x^2 - 14x - 9y^2 + 72y &= 200 && \text{group terms; add 200} \\
 7(x^2 - 2x + \underline{\quad}) - 9(y^2 - 8y + \underline{\quad}) &= 200 && \text{factor out leading coefficients} \\
 7(x^2 - 2x + 1) - 9(y^2 - 8y + 16) &= 200 + 7 + (-144) && \text{complete the square} \\
 \uparrow \qquad \qquad \qquad \uparrow & \qquad \qquad \qquad \uparrow && \text{→ add 7 + (-144)} \\
 \text{adds } 7(1) = 7 & \qquad \qquad \text{adds } -9(16) = -144 && \text{to right-hand side} \\
 7(x - 1)^2 - 9(y - 4)^2 &= 63 && \text{factored form} \\
 \frac{(x - 1)^2}{9} - \frac{(y - 4)^2}{7} &= 1 && \text{divide by 63 and simplify} \\
 \frac{(x - 1)^2}{3^2} - \frac{(y - 4)^2}{(\sqrt{7})^2} &= 1 && \text{write denominators in squared form}
 \end{aligned}$$

This is a horizontal hyperbola with $a = 3$ ($a^2 = 9$) and $b = \sqrt{7}$ ($b^2 = 7$). The center is at $(1, 4)$, with vertices $(-2, 4)$ and $(4, 4)$. Using the foci formula $c^2 = a^2 + b^2$ yields $c^2 = 9 + 7 = 16$, showing the foci are $(-3, 4)$ and $(5, 4)$ (4 units from center). The central rectangle is $2\sqrt{7} \approx 5.29$ by $2(3) = 6$.

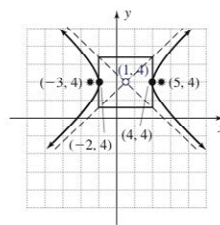
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10-29

Section 10.3 The Hyperbola

947

Drawing the rectangle and sketching the asymptotes to complete the graph, results in the graph shown.



Horizontal hyperbola

- Center at (1, 4)
- Vertices at (-2, 4) and (4, 4)
- Transverse axis: $y = 4$
- Conjugate axis: $x = 1$
- Location of foci: (-3, 4) and (5, 4)
- Width of rectangle (horizontal dimension and distance between vertices)
 $2a = 2(3) = 6$
- Length of rectangle (vertical dimension)
 $2b = 2(\sqrt{7}) \approx 5.29$

C. You've just learned how to locate the foci of a hyperbola and use the foci and other features to write its equation

Now try Exercises 61 through 70 ►

As with the ellipse, if any two of the values for a , b , and c are known, the relationship between them can be used to construct the equation of the hyperbola. See Exercises 71 through 78.

D. Applications Involving Foci

Applications involving the foci of a conic section can take many forms. As before, only partial information about the hyperbola may be available, and we'll determine a solution by manipulating a given equation, or constructing an equation from given facts.

EXAMPLE 7 ► Applying the Properties of a Hyperbola—The Path of a Comet

Comets with a high velocity cannot be captured by the Sun's gravity, and are slung around the Sun in a hyperbolic path with the Sun at one focus. If the path illustrated by the graph shown is modeled by the equation $2116x^2 - 400y^2 = 846,400$, how close did the comet get to the Sun? Assume units are in millions of miles and round to the nearest million.



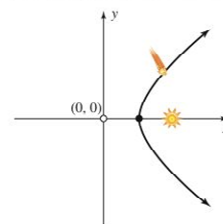
Solution ► We are essentially asked to find the distance between a vertex and focus. Begin by writing the equation in standard form:

$$2116x^2 - 400y^2 = 846,400 \quad \text{given}$$

$$\frac{x^2}{400} - \frac{y^2}{2116} = 1 \quad \text{divide by 846,400}$$

$$\frac{x^2}{20^2} - \frac{y^2}{46^2} = 1 \quad \text{write denominators in squared form}$$

This is a horizontal hyperbola with $a = 20$ ($a^2 = 400$) and $b = 46$ ($b^2 = 2116$). Use the foci formula to find c^2 and c .



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$$\begin{aligned} c^2 &= a^2 + b^2 \\ c^2 &= 400 + 2116 \\ c^2 &= 2516 \\ c &\approx -50 \text{ and } c \approx 50 \end{aligned}$$

Since $a = 20$ and $|c| \approx 50$, the comet came within $50 - 20 = 30$ million miles of the Sun.

Now try Exercises 81 through 84 ▶

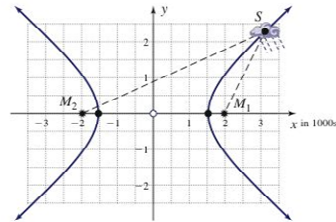
EXAMPLE 8 ▶ Applying the Properties of a Hyperbola—The Location of a Storm

Two amateur meteorologists, living 4 km apart (4000 m), see a storm approaching. The one farthest from the storm hears a loud clap of thunder 9 sec after the one nearest. Assuming the speed of sound is 340 m/sec, determine an equation that models possible locations for the storm at this time.



Solution ▶ Let M_1 represent the meteorologist nearest the storm and M_2 the farthest. Since M_2 heard the thunder 9 sec after M_1 , M_2 must be $9 \cdot 340 = 3060$ m farther away from the storm S . In other words, $|M_2S| - |M_1S| = 3060$. The set of all points that satisfy this description fit the definition of a hyperbola, and we'll use this fact to develop an equation model for possible locations of the storm. Let's place the information on a coordinate grid. For convenience, we'll use the straight line distance between M_1 and M_2 as the x -axis, with the origin an equal distance from each. With the constant difference equal to 3060, we have $2a = 3060$, $a = 1530$ from the definition of a hyperbola,

giving $\frac{x^2}{1530^2} - \frac{y^2}{b^2} = 1$. With $c = 2000$ m (the distance from the origin to M_1 or M_2), we find the value of b using the equation $c^2 = a^2 + b^2$: $2000^2 = 1530^2 + b^2$ or $b^2 = (2000)^2 - (1530)^2 = 1,659,100 \approx 1288^2$. The equation that models possible locations of the storm is



✓ **D.** You've just learned how to solve applications involving foci

Now try Exercises 85 and 86 ▶

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TECHNOLOGY HIGHLIGHT

Studying Hyperbolas

As with the circle and ellipse, the hyperbola must also be defined in two pieces in order to use a graphing calculator to study its graph. Consider the relation $4x^2 - 9y^2 = 36$. From our work in this section, we know this is the equation of a horizontal hyperbola centered at (0, 0). Solving for y gives


$$4x^2 - 9y^2 = 36 \quad \text{original equation}$$

$$-9y^2 = 36 - 4x^2 \quad \text{isolate } y^2\text{-term}$$

$$y^2 = \frac{36 - 4x^2}{-9} \quad \text{divide by } -9$$

$$y = \pm \sqrt{\frac{36 - 4x^2}{-9}} \quad \text{solve for } y$$

We can again separate this result into two parts: $Y_1 = \sqrt{\frac{36 - 4x^2}{-9}}$ gives the "upper half" of the hyperbola, and $Y_2 = -\sqrt{\frac{36 - 4x^2}{-9}}$ gives the "lower half." In Figure 10.24, note the use of parentheses on the **Y=** screen to ensure we're taking the square root of the entire expression.



Entering these on the **Y=** screen, graphing them with the window shown, and pressing the **TRACE** key gives the graph shown in Figure 10.25. Note the location of the cursor at $x = 0$, but no y -value is displayed. This is because the hyperbola is not defined at $x = 0$. Press the right arrow key **▶** and walk the cursor to the right until the y -values begin appearing. In fact, they begin to appear at (3, 0), which is one of the vertices of the hyperbola. We could also graph the asymptotes ($y = \pm \frac{2}{3}x$) by entering the lines as Y_3 and Y_4 on the **Y=** screen. The resulting graph is shown in Figure 10.26 using the standard window (the **TRACE** key has been pushed and the down arrow used to highlight Y_2). Use these ideas to complete the following exercises.

Figure 10.25

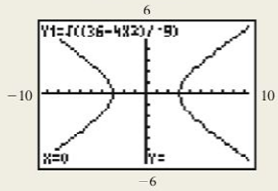
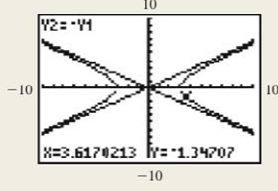


Figure 10.26



Exercise 1: Graph the hyperbola $25y^2 - 4x^2 = 100$ using a friendly window. What are the coordinates of the vertices? Use the **TRACE** feature to find the value(s) of y when $x = 4$. Determine (from the graph) the value(s) of y when $x = -4$, then verify your response using the **TABLE** feature.

Exercise 2: Graph the hyperbola $9x^2 - 16y^2 = 144$ using the standard window. Then determine the equations of the asymptotes and graph these as well. Why do the asymptotes intersect at the origin? When will the asymptotes *not* intersect at the origin?

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10.3 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

- The line that passes through the vertices of a hyperbola is called the _____ axis.
- The conjugate axis is _____ to the _____ axis and contains the _____ of the hyperbola.
- The center of a hyperbola is located _____ between the vertices.

4. The center of the hyperbola defined by $\frac{(x - 2)^2}{4^2} - \frac{(y - 3)^2}{5^2} = 1$ is at _____.

- Compare/Contrast the two methods used to find the asymptotes of a hyperbola. Include an example illustrating both methods.
- Explore/Explain why $A(x - h)^2 - B(y - k)^2 = F$ results in a hyperbola regardless of whether $A = B$ or $A \neq B$. Illustrate with an example.

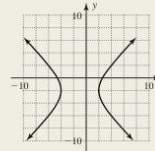
► DEVELOPING YOUR SKILLS

Graph each hyperbola. Label the center, vertices, and any additional points used.

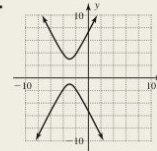
- | | |
|---|---|
| 7. $\frac{x^2}{9} - \frac{y^2}{4} = 1$ | 8. $\frac{x^2}{16} - \frac{y^2}{9} = 1$ |
| 9. $\frac{x^2}{4} - \frac{y^2}{9} = 1$ | 10. $\frac{x^2}{25} - \frac{y^2}{16} = 1$ |
| 11. $\frac{x^2}{49} - \frac{y^2}{16} = 1$ | 12. $\frac{x^2}{25} - \frac{y^2}{9} = 1$ |
| 13. $\frac{x^2}{36} - \frac{y^2}{16} = 1$ | 14. $\frac{x^2}{81} - \frac{y^2}{16} = 1$ |
| 15. $\frac{y^2}{9} - \frac{x^2}{1} = 1$ | 16. $\frac{y^2}{1} - \frac{x^2}{4} = 1$ |
| 17. $\frac{y^2}{12} - \frac{x^2}{4} = 1$ | 18. $\frac{y^2}{9} - \frac{x^2}{18} = 1$ |
| 19. $\frac{y^2}{9} - \frac{x^2}{9} = 1$ | 20. $\frac{y^2}{4} - \frac{x^2}{4} = 1$ |
| 21. $\frac{y^2}{36} - \frac{x^2}{25} = 1$ | 22. $\frac{y^2}{16} - \frac{x^2}{4} = 1$ |

For the graphs given, state the location of the vertices and the equation of the transverse axis. Then identify the location of the center and the equation of the conjugate axis. Note the scale used on each axis.

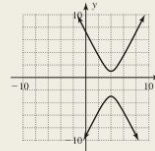
23.



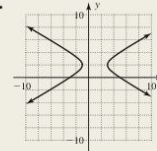
24.



25.



26.



Sketch a complete graph of each equation, including the asymptotes. Be sure to identify the center and vertices.

- $\frac{(y + 1)^2}{4} - \frac{x^2}{25} = 1$
- $\frac{y^2}{4} - \frac{(x - 2)^2}{9} = 1$
- $\frac{(x - 3)^2}{36} - \frac{(y + 2)^2}{49} = 1$
- $\frac{(x - 2)^2}{9} - \frac{(y - 1)^2}{4} = 1$

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10-33

Section 10.3 The Hyperbola

951

31. $\frac{(y + 1)^2}{7} - \frac{(x + 5)^2}{9} = 1$

32. $\frac{(y - 3)^2}{16} - \frac{(x + 2)^2}{5} = 1$

33. $(x - 2)^2 - 4(y + 1)^2 = 16$

34. $9(x + 1)^2 - (y - 3)^2 = 81$

35. $2(y + 3)^2 - 5(x - 1)^2 = 50$

36. $9(y - 4)^2 - 5(x - 3)^2 = 45$

37. $12(x - 4)^2 - 5(y - 3)^2 = 60$

38. $8(x - 4)^2 - 3(y - 3)^2 = 24$

39. $16x^2 - 9y^2 = 144$

40. $16x^2 - 25y^2 = 400$

41. $9y^2 - 4x^2 = 36$

42. $25y^2 - 4x^2 = 100$

43. $12x^2 - 9y^2 = 72$

44. $36x^2 - 20y^2 = 180$

45. $4x^2 - y^2 + 40x - 4y + 60 = 0$

46. $x^2 - 4y^2 - 12x - 16y + 16 = 0$

47. $x^2 - 4y^2 - 24y - 4x - 36 = 0$

48. $-9x^2 + 4y^2 - 18x - 24y - 9 = 0$

Classify each equation as that of a circle, ellipse, or hyperbola. Justify your response.

49. $-4x^2 - 4y^2 = -24$

50. $9y^2 = -4x^2 + 36$

51. $x^2 + y^2 = 2x + 4y + 4$

52. $x^2 = y^2 + 6y - 7$

53. $2x^2 - 4y^2 = 8$

54. $36x^2 + 25y^2 = 900$

55. $x^2 + 5 = 2y^2$

56. $x + y^2 = 3x^2 + 9$

57. $2x^2 = -2y^2 + x + 20$

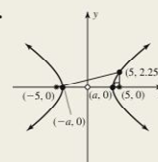
58. $2y^2 + 3 = 6x^2 + 8$

59. $16x^2 + 5y^2 - 3x + 4y = 538$

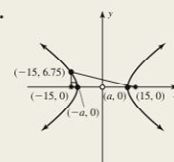
60. $9x^2 + 9y^2 - 9x + 12y + 4 = 0$

Use the definition of a hyperbola to find the distance between the vertices and the dimensions of the rectangle centered at (h, k) . Figures are not drawn to scale. Note that Exercises 63 and 64 are vertical hyperbolas.

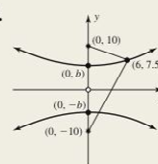
61.



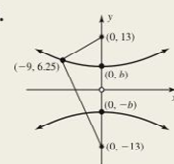
62.



63.



64.



Find and list the coordinates of the (a) center, (b) vertices, (c) foci, and (d) dimensions of the central rectangle. Then (e) sketch the graph, including the asymptotes.

65. $4x^2 - 9y^2 - 24x + 72y - 144 = 0$

66. $4x^2 - 36y^2 - 40x + 144y - 188 = 0$

67. $16x^2 - 4y^2 + 24y - 100 = 0$

68. $81x^2 - 162x - 4y^2 - 243 = 0$

69. $9x^2 - 3y^2 - 54x - 12y + 33 = 0$

70. $10x^2 + 60x - 5y^2 + 20y - 20 = 0$

Find the equation of the hyperbola (in standard form) that satisfies the following conditions:

71. vertices at $(-6, 0)$ and $(6, 0)$; foci at $(-8, 0)$ and $(8, 0)$

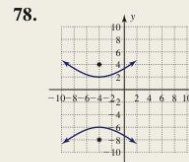
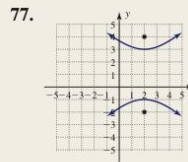
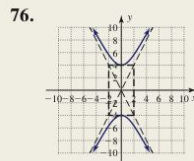
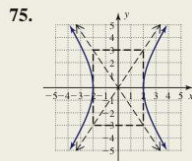
72. vertices at $(-4, 0)$ and $(4, 0)$; foci at $(-6, 0)$ and $(6, 0)$

73. foci at $(-2, -3\sqrt{2})$ and $(-2, 3\sqrt{2})$; length of conjugate axis: 6 units

74. foci at $(-5, 2)$ and $(7, 2)$; length of conjugate axis: 8 units

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Use the characteristics of a hyperbola and the graph given to write the related equation and state the location of the foci (75 and 76) or the dimensions of the central rectangle (77 and 78).

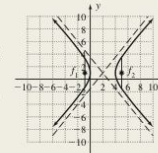


▶ WORKING WITH FORMULAS

79. Equation of a semi-hyperbola: $y = \sqrt{\frac{36 - 4x^2}{-9}}$
 The “upper half” of a certain hyperbola is given by the equation shown. (a) Simplify the radicand, (b) state the domain of the expression, and (c) enter the expression as Y_1 on a graphing calculator and graph. What is the equation for the “lower half” of this hyperbola?

80. Focal chord of a hyperbola: $L = \frac{2b^2}{a}$
 The focal chords of a hyperbola are line segments parallel to the conjugate axis with endpoints on the

hyperbola, and containing certain points f_1 and f_2 called the foci (see grid). The length of the chord is given by the formula shown. Use it to find the length of the focal chord for the hyperbola indicated, then compare the calculated value with the length estimated from the given graph:



$$\frac{(x - 2)^2}{4} - \frac{(y - 1)^2}{5} = 1.$$

▶ APPLICATIONS

- 81. **Stunt pilots:** At an air show, a stunt plane dives along a hyperbolic path whose vertex is directly over the grandstands. If the plane's flight path can be modeled by the hyperbola $25y^2 - 1600x^2 = 40,000$, what is the minimum altitude of the plane as it passes over the stands? Assume x and y are in yards.
- 82. **Flying clubs:** To test their skill as pilots, the members of a flight club attempt to drop sandbags on a target placed in an open field, by diving along a hyperbolic path whose vertex is directly over the target area. If the flight path of the plane flown by the club's president is modeled by $9y^2 - 16x^2 = 14,400$, what is the minimum altitude of her plane as it passes over the target? Assume x and y are in feet.

83. **Nuclear cooling towers:** The natural draft cooling towers for nuclear power stations are called *hyperboloids of one sheet*. The



- perpendicular cross sections of these hyperboloids form two branches of a hyperbola. Suppose the central cross section of one such tower is modeled by the hyperbola $1600x^2 - 400(y - 50)^2 = 640,000$. What is the minimum distance between the sides of the tower? Assume x and y are in feet.
- 84. **Charged particles:** It has been shown that when like particles with a common charge are hurled at each other, they deflect and travel along paths that are hyperbolic. Suppose the path of two such particles is modeled by the hyperbola $x^2 - 9y^2 = 36$. What is the minimum distance between the particles as they approach each other? Assume x and y are in microns.
- 85. **Locating a ship using radar:** Under certain conditions, the properties of a hyperbola can be used to help locate the position of a ship. Suppose two radio stations are located 100 km apart along a straight shoreline. A ship is sailing parallel to the shore and is 60 km out to sea. The ship sends out a distress call that is picked up by the closer station

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10-35

Section 10.3 The Hyperbola

953

in 0.4 milliseconds (msec—one-thousandth of a second), while it takes 0.5 msec to reach the station that is farther away. Radio waves travel at a speed of approximately 300 km/msec. Use this information to find the equation of a hyperbola that will help you find the location of the ship, then find the coordinates of the ship. (*Hint:* Draw the hyperbola on a coordinate system with the radio stations on the x -axis at the foci, then use the definition of a hyperbola.)

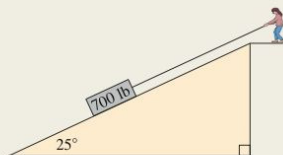
86. Locating a plane using radar: Two radio stations are located 80 km apart along a straight shoreline, when a “mayday” call (a plea for immediate help) is received from a plane that is about to ditch in the ocean (attempt a water landing). The plane was flying at low altitude, parallel to the shoreline, and 20 km out when it ran into trouble. The plane’s distress call is picked up by the closer station in 0.1 msec, while it takes 0.3 msec to reach the other. Use this information to construct the equation of a hyperbola that will help you find the location of the ditched plane, then find the coordinates of the plane. Also see Exercise 85.

87. It is possible for the plane to intersect only the vertex of the cone or to be tangent to the sides. These are called **degenerate cases** of a conic section. Many times we’re unable to tell if the equation represents a degenerate case until it’s written in standard form. Write the following equations in standard form and comment.

- a. $4x^2 - 32x - y^2 + 4y + 60 = 0$
- b. $x^2 - 4x + 5y^2 - 40y + 84 = 0$

► MAINTAINING YOUR SKILLS

93. (7.3) In weight-lifting competitions, Ursula Unger has shown she can lift up to 350 lb. Use a vector analysis to determine whether she will be able to pull the crate up the frictionless ramp shown.

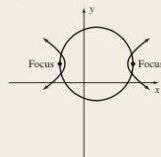


94. (5.1) The wheels on a motorized scooter are rotating at 403 rpm. If the wheels have a 2.5 in. radius, how fast is the scooter traveling in miles per hour?

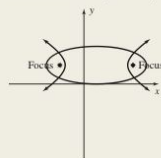
88. For a greater understanding as to *why* the branches of a hyperbola are asymptotic, solve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for y , then consider what happens as $x \rightarrow \infty$ (note that $x^2 - k \approx x^2$ for large x).

89. Which has a greater area: (a) The central rectangle of the hyperbola given by $(x - 5)^2 - (y + 4)^2 = 57$, (b) the circle given by $(x - 5)^2 + (y + 4)^2 = 57$, or (c) the ellipse given by $9(x - 5)^2 + 10(y + 4)^2 = 570$?

90. Find the equation of the circle shown, given the equation of the hyperbola: $9(x - 2)^2 - 25(y - 3)^2 = 225$



91. Find the equation of the ellipse shown, given the equation of the hyperbola and $(2, 0)$ is on the graph of the ellipse. The hyperbola and ellipse share the same foci: $9(x - 2)^2 - 25(y - 3)^2 = 225$



92. Verify that for a central hyperbola, a circle that circumscribes the central rectangle must also go through both foci.

95. (1.4) The number $z = 1 + i\sqrt{2}$ is a solution to two out of the three equations given. Which two?

- a. $x^4 + 4 = 0$
- b. $x^3 - 6x^2 + 11x - 12 = 0$
- c. $x^2 - 2x + 3 = 0$

96. (5.4) A government-approved company is licensed to haul toxic waste. Each container of solid waste weighs 800 lb and has a volume of 100 ft³. Each container of liquid waste weighs 1000 lb and is 60 ft³ in volume. The revenue from hauling solid waste is \$300 per container, while the revenue from liquid waste is \$375 per container. The truck used by this company has a weight capacity of 39.8 tons and a volume capacity of 6960 ft³. What combination of solid and liquid weight containers will produce the maximum revenue?

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10.4 The Analytic Parabola

Learning Objectives

In Section 10.4 you will learn how to:

- A. Graph parabolas with a horizontal axis of symmetry
- B. Identify and use the focus-directrix form of the equation of a parabola
- C. Solve nonlinear systems involving the conic sections
- D. Solve applications of the analytic parabola

In previous coursework, you likely learned that the graph of a quadratic function was a parabola. Parabolas are actually the fourth and final member of the family of conic sections, and as we saw in Section 10.1, the graph can be obtained by observing the intersection of a plane and a cone. If the plane is parallel to the generator of the cone (shown as a dark line in Figure 10.27), the intersection of the plane with one nappe forms a parabola. In this section we develop the general equation of a parabola from its analytic definition, opening a new realm of applications that extends far beyond those involving only zeroes and extreme values.

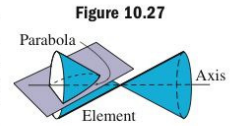


Figure 10.27

A. Parabolas with a Horizontal Axis

An introductory study of parabolas generally involves those with a vertical axis, defined by the equation $y = ax^2 + bx + c$. Unlike the previous conic sections, this equation has *only one second-degree (squared) term in x* and defines a function. As a review of our work in section 3.1, the primary characteristics are listed here and illustrated in Figure 10.28.

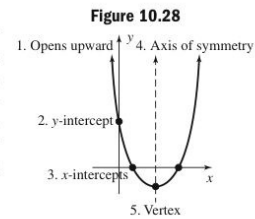


Figure 10.28

Vertical Parabolas

For a second-degree equation of the form $y = ax^2 + bx + c$, the graph is a vertical parabola with these characteristics:

- | | |
|--|---|
| 1. opens upward if $a > 0$, downward if $a < 0$. | 2. y-intercept: $(0, c)$
(substitute 0 for x) |
| 3. x-intercept(s): substitute 0 for y and solve. | 4. axis of symmetry: $x = \frac{-b}{2a}$ |
| 5. vertex: $\left(\frac{-b}{2a}, y\right)$ | |

See Exercises 7 through 12 for additional review and practice.

Horizontal Parabolas

Similar to our study of horizontal and vertical hyperbolas, the graph of a parabola can open to the right or left, as well as up or down. After interchanging the variables x and y in the standard equation, we obtain the parabola $x = ay^2 + by + c$, noting the resulting graph will be a reflection about the line $y = k$. Here, the axis of symmetry is a horizontal line and factoring or the quadratic formula is used to find the *y-intercepts* (if they exist). Note that although the graph is still a parabola—it is *not the graph of a function*.

Horizontal Parabolas

For a second-degree equation of the form $x = ay^2 + by + c$, the graph is a horizontal parabola with these characteristics:

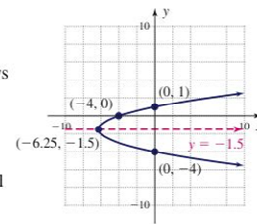
- | | |
|--|--|
| 1. opens right if $a > 0$, left if $a < 0$. | 2. x-intercept: $(c, 0)$ (substitute 0 for y) |
| 3. y-intercept(s): substitute 0 for x and solve. | 4. axis of symmetry: $y = \frac{-b}{2a}$ |
| 5. vertex: $\left(x, \frac{-b}{2a}\right)$ | |

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EXAMPLE 1 ▶ Graphing a Horizontal Parabola

Graph the relation whose equation is $x = y^2 + 3y - 4$, then state the domain and range of the relation.

Solution ▶ Since the equation has a single squared term in y , the graph will be a horizontal parabola. With $a > 0$ ($a = 1$), the parabola opens to the right. The x -intercept is $(-4, 0)$. Factoring shows the y -intercepts are $y = -4$ and $y = 1$. The axis of symmetry is $y = \frac{-3}{2} = -1.5$, and substituting this value into the original equation gives $x = -6.25$. The coordinates of the vertex are $(-6.25, -1.5)$. Using horizontal and vertical boundary lines we find the domain for this relation is $x \in [-6.25, \infty)$ and the range is $y \in (-\infty, \infty)$. The graph is shown.



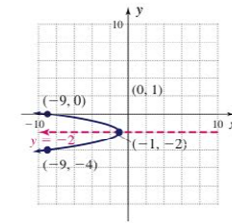
Now try Exercises 13 through 18 ▶

As with the vertical parabola, the equation of a horizontal parabola can be written as a transformation: $x = a(y \pm k)^2 \pm h$ by completing the square. Note that in this case, the vertical shift is k units *opposite the sign*, with a horizontal shift of h units in the same direction as the sign.

EXAMPLE 2 ▶ Graphing a Horizontal Parabola by Completing the Square

Graph by completing the square: $x = -2y^2 - 8y - 9$.

Solution ▶ Using the original equation, we note the graph will be a horizontal parabola opening to the left ($a = -2$) and have an x -intercept of $(-9, 0)$. Completing the square gives $x = -2(y^2 + 4y + 4) - 9 + 8$, so $x = -2(y + 2)^2 - 1$. The vertex is at $(-1, -2)$ and $y = -2$ is the axis of symmetry. This means there are no y -intercepts, a fact that comes to light when we attempt to solve the equation after substituting 0 for x :



$$-2(y + 2)^2 - 1 = 0 \quad \text{substitute 0 for } x$$

$$(y + 2)^2 = -\frac{1}{2} \quad \text{isolate squared term}$$

A. You've just learned how to graph parabolas with a horizontal axis of symmetry

The equation has no real roots. Using symmetry, the point $(-9, -4)$ is also on the graph. After plotting these points we obtain the graph shown.

Now try Exercises 19 through 36 ▶

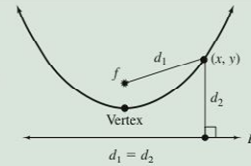
B. The Focus-Directrix Form of the Equation of a Parabola

As with the ellipse and hyperbola, many significant applications of the parabola rely on its analytical definition rather than its algebraic form. From the construction of radio telescopes to the manufacture of flashlights, the location of the focus of a parabola is critical. To understand these and other applications, we use the analytic definition of a parabola first introduced in Section 10.1.

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Definition of a Parabola

Given a fixed point f and fixed line D in the plane, a parabola is the set of all points (x, y) such that the distance from f to (x, y) is equal to the distance from line D to (x, y) . The fixed point f is the **focus** of the parabola, and the fixed line is the **directrix**.



WORTHY OF NOTE

For the analytic parabola, we use p to designate the focus, since c is so commonly used as the constant term in $y = ax^2 + bx + c$.

The general equation of a parabola can be obtained by combining this definition with the distance formula. With no loss of generality, we can assume the parabola shown in the definition box is oriented in the plane with the vertex at $(0, 0)$ and the focus at $(0, p)$. As the diagram in Figure 10.29 indicates, this gives the directrix an equation of $y = -p$ with all points on D having coordinates of $(x, -p)$.

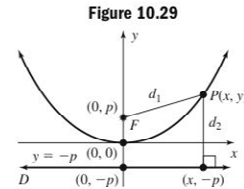


Figure 10.29

Using $d_1 = d_2$ the distance formula yields

$$\begin{aligned} \sqrt{(x-0)^2 + (y-p)^2} &= \sqrt{(x-x)^2 + (y+p)^2} && \text{from definition} \\ (x-0)^2 + (y-p)^2 &= (x-x)^2 + (y+p)^2 && \text{square both sides} \\ x^2 + y^2 - 2py + p^2 &= 0 + y^2 + 2py + p^2 && \text{simplify; expand binomials} \\ x^2 - 2py &= 2py && \text{subtract } p^2 \text{ and } y^2 \\ x^2 &= 4py && \text{isolate } x^2 \end{aligned}$$

The resulting equation is called the **focus-directrix form** of a *vertical parabola* with vertex at $(0, 0)$. If we had begun by orienting the parabola so it opened to the right, we would have obtained the equation of a *horizontal parabola* with vertex $(0, 0)$: $y^2 = 4px$.

The Equation of a Parabola in Focus-Directrix Form

Vertical Parabola

$$x^2 = 4py$$

focus $(0, p)$, directrix: $y = -p$
 If $p > 0$, opens upward.
 If $p < 0$, opens downward.

Horizontal Parabola

$$y^2 = 4px$$

focus at $(p, 0)$ directrix: $x = -p$
 If $p > 0$, opens to the right.
 If $p < 0$, opens to the left.

For a parabola, note there is only one second-degree term.

EXAMPLE 3 ▶ **Locating the Focus and Directrix of a Parabola**

Find the vertex, focus, and directrix for the parabola defined by $x^2 = -12y$. Then sketch the graph, including the focus and directrix.

Solution ▶ Since the x -term is squared and no shifts have been applied, the graph will be a vertical parabola with a vertex of $(0, 0)$. Use a direct comparison between the given equation and the focus-directrix form to determine the value of p :

$$\begin{aligned} x^2 &= -12y && \text{given equation} \\ &\downarrow && \\ x^2 &= 4y && \text{focus-directrix form} \end{aligned}$$

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10-39

Section 10.4 The Analytic Parabola

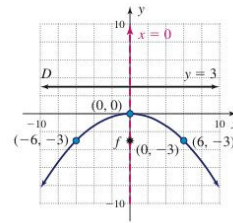
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This shows:

$$4p = -12$$

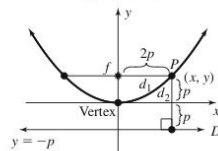
$$p = -3$$

Since $p = -3$ ($p < 0$), the parabola opens downward, with the focus at $(0, -3)$ and directrix $y = 3$. To complete the graph we need a few additional points. Since 36 (6^2) is divisible by 12 , we can use inputs of $x = 6$ and $x = -6$, giving the points $(6, -3)$ and $(-6, -3)$. Note the axis of symmetry is $x = 0$. The graph is shown.



Now try Exercises 37 through 48 ▶

Figure 10.30



As an alternative to calculating additional points to sketch the graph, we can use what is called the **focal chord** of the parabola. Similar to the ellipse and hyperbola, the focal chord is a line segment that contains the focus, is parallel to the directrix, and has its endpoints on the graph. Using the definition of a parabola and the diagram in Figure 10.30, we see the horizontal distance from f to (x, y) is $2p$. Since $d_1 = d_2$, a line segment parallel to the directrix from the focus to the graph will also have a length of $|2p|$, and the focal chord of any parabola has a total length of $|4p|$. Note that in Example 3, the points we happened to choose were actually the endpoints of the focal chord.

Finally, if the vertex of a vertical parabola is shifted to (h, k) , the equation will have the form $(x \pm h)^2 = 4p(y \pm k)$. As with the other conic sections, both the horizontal and vertical shifts are “opposite the sign.”

EXAMPLE 4 ▶ Locating the Focus and Directrix of a Parabola

Find the vertex, focus, and directrix for the parabola whose equation is given, then sketch the graph, including the focus and directrix: $x^2 - 6x + 12y - 15 = 0$.

Solution ▶ Since only the x -term is squared, the graph will be a vertical parabola. To find the concavity, vertex, focus, and directrix, we complete the square in x and use a direct comparison between the shifted form and the focus-directrix form:

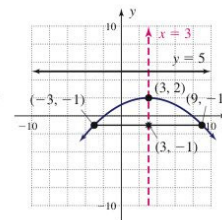
$$x^2 - 6x + 12y - 15 = 0 \quad \text{given equation}$$

$$x^2 - 6x + \underline{\quad} = -12y + 15 \quad \text{complete the square in } x$$

$$x^2 - 6x + 9 = -12y + 24 \quad \text{add 9}$$

$$(x - 3)^2 = -12(y - 2) \quad \text{factor}$$

Notice the parabola has been shifted 3 right and 2 up, so all features of the parabola will likewise be shifted. Since we have $4p = -12$ (the coefficient of the linear term), we know $p = -3$ ($p < 0$) and the parabola opens downward. If the parabola were in standard position, the vertex would be at $(0, 0)$, the focus at $(0, -3)$ and the directrix a horizontal line at $y = 3$. But since the parabola is shifted 3 right and 2 up, we add 3 to all x -values and 2 to all y -values to locate the features of the shifted parabola. The vertex is at



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$(0 + 3, 0 + 2) = (3, 2)$. The focus is $(0 + 3, -3 + 2) = (3, -1)$ and the directrix is $y = 3 + 2 = 5$. Finally, the horizontal distance from the focus to the graph is $|2p| = 6$ units (since $|4p| = 12$), giving us the additional points $(-3, -1)$ and $(9, -1)$. The graph is shown.

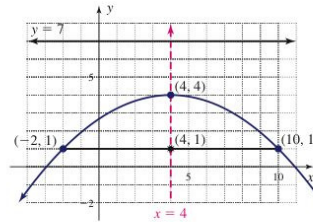
Now try Exercises 49 through 60 ▶

In many cases, we need to construct the equation of the parabola when only partial information is known, as illustrated in Example 5.

EXAMPLE 5 ▶ Constructing the Equation of a Parabola

Find the equation of the parabola with vertex $(4, 4)$ and focus $(4, 1)$. Then graph the parabola using the equation and focal chord.

Solution ▶ As the vertex and focus are on a vertical line, we have a vertical parabola with general equation $(x \pm h)^2 = 4p(y \pm k)$. The distance p from vertex to focus is 3 units, and with the focus below the vertex, the parabola opens downward so $p = -3$. Using the focal chord, the horizontal distance from $(4, 1)$ to the graph is $|2p| = |2(-3)| = 6$, giving points $(-2, 1)$ and $(10, 1)$. The vertex is shifted 4 units right and 4 units up from $(0, 0)$, showing $h = 4$ and $k = 4$, and the equation of the parabola must be $(x - 4)^2 = -12(y - 4)$, with directrix $y = 7$. The graph is shown.



B. You've just learned how to identify and use the focus-directrix form of the equation of a parabola

Now try Exercises 61 through 76 ▶

C. Nonlinear Systems and the Conic Sections

Similar to our work with nonlinear systems in Section 5.3, the graphing, substitution, or elimination method can still be used when the system involves a conic section. When both equations in the system have at least one second degree term, it is generally easier to use the elimination method.

EXAMPLE 6 ▶ Solving a System of Nonlinear Equations

Solve the system using elimination:
$$\begin{cases} 2y^2 - 5x^2 = 13 \\ 3x^2 + 4y^2 = 39 \end{cases}$$

Solution ▶ The first equation represents a vertical and central hyperbola, while the second represents a horizontal and central ellipse. After writing the system with the x - and y -terms in the same order, we obtain
$$\begin{cases} -5x^2 + 2y^2 = 13 \\ 3x^2 + 4y^2 = 39 \end{cases}$$
 Using $-2R1 + R2$ will eliminate the y -term.

$$\begin{array}{rcl} \begin{cases} 10x^2 - 4y^2 = -26 & -2R1 \\ 3x^2 + 4y^2 = 39 & +R2 \\ \hline 13x^2 + 0 = 13 & \text{sum} \\ x^2 = 1 & \text{divide by 13} \\ x = -1 \text{ or } x = 1 & \text{square root property} \end{cases} \end{array}$$

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10-41

Section 10.4 The Analytic Parabola

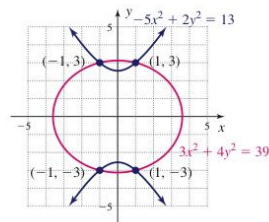
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Substituting $x = 1$ and $x = -1$ into the second equation we obtain

$$\begin{aligned} 3(1)^2 + 4y^2 &= 39 \\ 3 + 4y^2 &= 39 \\ 4y^2 &= 36 \\ y^2 &= 9 \\ y &= -3 \text{ or } y = 3 \end{aligned}$$

$$\begin{aligned} 3(-1)^2 + 4y^2 &= 39 \\ 3 + 4y^2 &= 39 \\ 4y^2 &= 36 \\ y^2 &= 9 \\ y &= -3 \text{ or } y = 3 \end{aligned}$$

Since -1 and 1 each generated *two outputs*, there are a total of four ordered pair solutions: $(1, -3)$, $(1, 3)$, $(-1, -3)$, and $(-1, 3)$. The graph is shown and supports our results.



C. You've just learned how to solve nonlinear systems involving the conic sections

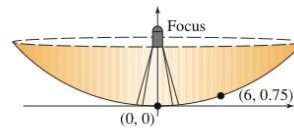
Now try Exercises 77 through 82 ►

D. Application of the Analytic Parabola

Here is just one of the many ways the analytic definition of a parabola can be applied. There are several others in the exercise set.

EXAMPLE 7 ► Locating the Focus of a Parabolic Receiver

The diagram shows the cross section of a radio antenna dish. Engineers have located a point on the cross section that is 0.75 m above and 6 m to the right of the vertex. At what coordinates should the engineers build the focus of the antenna?



Solution ► By inspection we see this is a vertical parabola with center at $(0, 0)$. This means its equation must be of the form $x^2 = 4py$. Because we know $(6, 0.75)$ is a point on this graph, we can substitute $(6, 0.75)$ in this equation and solve for p :

$$\begin{aligned} x^2 &= 4py && \text{equation for vertical parabola, vertex at } (0, 0) \\ (6)^2 &= 4p(0.75) && \text{substitute 6 for } x \text{ and } 0.75 \text{ for } y \\ 36 &= 3p && \text{simplify} \\ p &= 12 && \text{result} \end{aligned}$$

With $p = 12$, we see that the focus must be located at $(0, 12)$, or 12 m directly above the vertex.

Now try Exercises 85 through 92 ►

D. You've just learned how to solve applications of the analytic parabola

Note that in many cases, the focus of a parabolic dish may be taller than the rim of the dish.

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10.4 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

- The equation $x = ay^2 + by + c$ is that of a(n) _____ parabola, opening to the _____ if $a > 0$ and to the left if _____.
- If point P is on the graph of a parabola with directrix D , the distance from P to line D is equal to the distance between P and the _____ of the parabola.
- Given $y^2 = 4px$, the focus is at _____ and the equation of the directrix is _____.
- Given $x^2 = -16y$, the value of p is _____ and the coordinates of the focus are _____.
- Discuss/Explain how to find the vertex, directrix, and focus from the equation $(x - h)^2 = 4p(y - k)$.
- If a horizontal parabola has a vertex of $(2, -3)$ with $a > 0$, what can you say about the y -intercepts? Will the graph always have an x -intercept? Explain.

► DEVELOPING YOUR SKILLS

Find the x - and y -intercepts (if they exist) and the vertex of the parabola. Then sketch the graph by using symmetry and a few additional points or completing the square and shifting a parent function. Scale the axes as needed to comfortably fit the graph and state the domain and range.

- $y = x^2 - 2x - 3$
- $y = x^2 + 6x + 5$
- $y = 2x^2 - 8x - 10$
- $y = 3x^2 + 12x - 15$
- $y = 2x^2 + 5x - 7$
- $y = 2x^2 - 7x + 3$

Find the x - and y -intercepts (if they exist) and the vertex of the graph. Then sketch the graph using symmetry and a few additional points (scale the axes as needed). Finally, state the domain and range of the relation.

- $x = y^2 - 2y - 3$
- $x = y^2 - 4y - 12$
- $x = -y^2 + 6y + 7$
- $x = -y^2 + 8y - 12$
- $x = -y^2 + 8y - 16$
- $x = -y^2 + 6y - 9$

Sketch using symmetry and shifts of a basic function. Be sure to find the x - and y -intercepts (if they exist) and the vertex of the graph, then state the domain and range of the relation.

- $x = y^2 - 6y$
- $x = y^2 - 8y$
- $x = y^2 - 4$
- $x = y^2 - 9$
- $x = -y^2 + 2y - 1$
- $x = -y^2 + 4y - 4$
- $x = y^2 + y - 6$
- $x = y^2 + 4y - 5$

- $x = y^2 - 10y + 4$
- $x = y^2 + 12y - 5$
- $x = 3 - 8y - 2y^2$
- $x = 2 - 12y + 3y^2$
- $y = (x - 2)^2 + 3$
- $y = (x + 2)^2 - 4$
- $x = (y - 3)^2 + 2$
- $x = (y + 1)^2 - 4$
- $x = 2(y - 3)^2 + 1$
- $x = -2(y + 3)^2 - 5$

Find the vertex, focus, and directrix for the parabolas defined by the equations given, then use this information to sketch a complete graph (illustrate and name these features). For Exercises 43 to 60, also include the focal chord.

- $x^2 = 8y$
- $x^2 = 16y$
- $x^2 = -24y$
- $x^2 = -20y$
- $x^2 = 6y$
- $x^2 = 18y$
- $y^2 = -4x$
- $y^2 = -12x$
- $y^2 = 18x$
- $y^2 = 20x$
- $y^2 = -10x$
- $y^2 = -14x$
- $x^2 - 8x - 8y + 16 = 0$
- $x^2 - 10x - 12y + 25 = 0$
- $x^2 - 14x - 24y + 1 = 0$
- $x^2 - 10x - 12y + 1 = 0$
- $3x^2 - 24x - 12y + 12 = 0$
- $2x^2 - 8x - 16y - 24 = 0$

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10-43

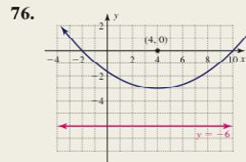
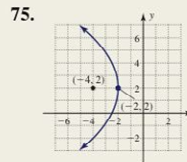
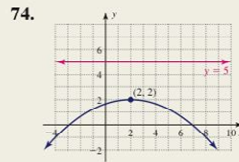
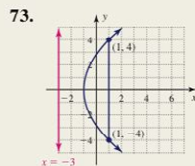
Section 10.4 The Analytic Parabola

- 55. $y^2 - 12y - 20x + 36 = 0$
- 56. $y^2 - 6y - 16x + 9 = 0$
- 57. $y^2 - 6y + 4x + 1 = 0$
- 58. $y^2 - 2y + 8x + 9 = 0$
- 59. $2y^2 - 20y + 8x + 2 = 0$
- 60. $3y^2 - 18y + 12x + 3 = 0$

For Exercises 61–72, find the equation of the parabola in standard form that satisfies the conditions given.

- | | |
|---------------------|----------------------|
| 61. focus: (0, 2) | 62. focus: (0, -3) |
| directrix: $y = -2$ | directrix: $y = 3$ |
| 63. focus: (4, 0) | 64. focus: (-3, 0) |
| directrix: $x = -4$ | directrix: $x = 3$ |
| 65. focus: (0, -5) | 66. focus: (5, 0) |
| directrix: $y = 5$ | directrix: $x = -5$ |
| 67. vertex: (2, -2) | 68. vertex: (4, 1) |
| focus: (-1, -2) | focus: (1, 1) |
| 69. vertex: (4, -7) | 70. vertex: (-3, -4) |
| focus: (4, -4) | focus: (-3, -1) |
| 71. focus: (3, 4) | 72. focus: (-1, 2) |
| directrix: $y = 0$ | directrix: $x = -5$ |

For the graphs in Exercises 73–76, only two of the following four features are displayed: vertex, focus, directrix, and endpoints of the focal chord. Find the remaining two features and the equation of the parabola.



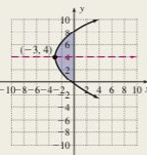
Solve using substitution or elimination, then graph the system.

- | | |
|---|--|
| 77. $\begin{cases} x^2 + y^2 = 25 \\ 2x^2 - 3y^2 = 5 \end{cases}$ | 78. $\begin{cases} y^2 - x^2 = 12 \\ x^2 + y^2 = 20 \end{cases}$ |
| 79. $\begin{cases} x^2 - y = 4 \\ y^2 - x^2 = 16 \end{cases}$ | 80. $\begin{cases} 2x^2 - 3y^2 = 38 \\ x^2 + 5y = 35 \end{cases}$ |
| 81. $\begin{cases} 5x^2 - 2y^2 = 75 \\ 2x^2 + 3y^2 = 125 \end{cases}$ | 82. $\begin{cases} 3x^2 - 7y^2 = 20 \\ 4x^2 + 9y^2 = 45 \end{cases}$ |

▶ WORKING WITH FORMULAS

83. The area of a right parabolic segment: $A = \frac{2}{3}ab$

A right parabolic segment is that part of a parabola formed by a line perpendicular to its axis, which cuts the parabola. The area of this segment is given by the formula shown, where b is the length of the chord cutting the parabola



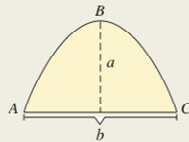
84. The arc length of a right parabolic segment:

$$\frac{1}{2}\sqrt{b^2 + 16a^2} + \frac{b^2}{8a} \ln\left(\frac{4a + \sqrt{b^2 + 16a^2}}{b}\right)$$

Although a fairly simple concept, finding the length of the parabolic arc traversed by a projectile requires a good deal of computation. To find the

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length of the arc ABC shown, we use the formula given where a is the maximum height attained by the projectile, b is the horizontal distance it traveled, and



“ln” represents the natural log function. Suppose a baseball thrown from centerfield reaches a maximum height of 20 ft and traverses an arc length of 340 ft. Will the ball reach the catcher 310 ft away without bouncing?

► APPLICATIONS

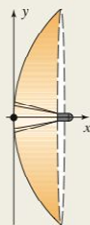
85. Parabolic car headlights: The cross section of a typical car headlight can be modeled by an equation similar to $25x = 16y^2$, where x and y are in inches and $x \in [0, 4]$. Use this information to graph the relation for the indicated domain.

86. Parabolic flashlights: The cross section of a typical flashlight reflector can be modeled by an equation similar to $4x = y^2$, where x and y are in centimeters and $x \in [0, 2.25]$. Use this information to graph the relation for the indicated domain.



87. Parabolic sound receivers: Sound technicians at professional sports events often use parabolic receivers as they move along the sidelines. If a two-dimensional cross section of the receiver is modeled by the equation $y^2 = 54x$, and is 36 in. in diameter, how deep is the parabolic receiver? What is the location of the focus? [Hint: Graph the parabola on the coordinate grid (scale the axes).]

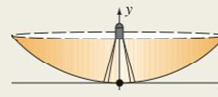
Exercise 87



88. Parabolic sound receivers: Private investigators will often use a smaller and less expensive

parabolic receiver (see Exercise 87) to gather information for their clients. If a two-dimensional cross section of the receiver is modeled by the equation $y^2 = 24x$, and the receiver is 12 in. in diameter, how deep is the parabolic dish? What is the location of the focus?

89. Parabolic radio wave receivers: The program known as S.E.T.I. (Search for Extra-Terrestrial Intelligence) identifies a group of scientists using radio telescopes to look for radio signals from possible intelligent species in outer space. The radio telescopes are actually parabolic dishes that vary in size from a few feet to hundreds of feet in diameter. If a particular radio telescope is 100 ft in diameter and has a cross section modeled by the equation $x^2 = 167y$, how deep is the parabolic dish? What is the location of the focus? [Hint: Graph the parabola on the coordinate grid (scale the axes).]



90. Solar furnace: Another form of technology that uses a parabolic dish is called a solar furnace. In general, the rays of the Sun are reflected by the dish and concentrated at the focus, producing extremely high temperatures. Suppose the dish of one of these parabolic reflectors had a 30-ft diameter and a cross section modeled by the equation $x^2 = 50y$. How deep is the parabolic dish? What is the location of the focus?



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10-45

Section 10.4 The Analytic Parabola

91. The reflector of a large, commercial flashlight has the shape of a parabolic dish, with a diameter of 10 cm and a depth of 5 cm. What equation will the engineers and technicians use for the manufacture of the dish? How far from the vertex (the lowest point of the dish) will the bulb be placed? (*Hint: Analyze the information using a coordinate system.*)

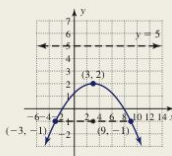
92. The reflector of an industrial spotlight has the shape of a parabolic dish with a diameter of 120 cm. What is the depth of the dish if the correct placement of the bulb is 11.25 cm above the vertex (the lowest point of the dish)? What equation will the engineers and technicians use for the manufacture of the dish? (*Hint: Analyze the information using a coordinate system.*)

► EXTENDING THE CONCEPT

93. In a study of quadratic graphs from the equation $y = ax^2 + bx + c$, no mention is made of a parabola's focus and directrix. Generally, when $a \geq 1$, the focus of a parabola is very near its vertex. Complete the square of the function $y = 2x^2 - 8x$ and write the result in the form $(x - h)^2 = 4p(y - k)$. What is the value of p ? What are the coordinates of the vertex?

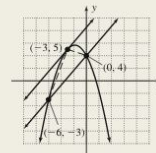
94. Like the ellipse and hyperbola, the focal chord of a parabola (also called the **latus rectum**) can be used to help sketch its graph. From our earlier work, we know the endpoints of the focal chord are $2p$ units from the focus. Write the equation $-12y + 15 = x^2 - 6x$ in the form $4p(y \pm k) = (x \pm h)^2$, and use the endpoints of the focal chord to help graph the parabola.

Exercise 94



95. In Exercise 83, a formula was given for the area of a right parabolic segment. The area of an *oblique* parabolic segment (the line segment cutting the parabola is *not* perpendicular to the axis) is more complex, as it involves locating the point where a line parallel to this segment is tangent (touches at only one point) to the parabola. The formula is $A = \frac{2}{3}T$, where T represents the area of the triangle formed by the endpoints of the segment and this point of tangency. What is the area of the parabolic segment shown (assuming the lines are parallel)? See Section 9.1, Exercises 46 and 47 and Section 9.4, Example 3.

Exercise 95



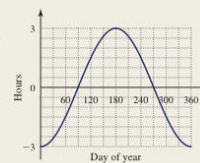
► MAINTAINING YOUR SKILLS

96. (6.6) Find all real solutions to $\sec \theta = 1.1547$ (round to the nearest degree).

97. (3.3/3.4) Use the function $f(x) = x^5 + 2x^4 + 17x^3 + 34x^2 - 18x - 36$ to comment and give illustrations of the tools available for working with polynomials: (a) synthetic division, (b) rational roots theorem, (c) the remainder and factor theorems, (d) the test for $x = -1$ and $x = 1$, (e) the upper/lower bounds property, (f) Descartes' rule of signs, and (g) roots of multiplicity (bounces, cuts, alternating intervals).

98. (1.6) Find all roots (real and complex) to the equation $x^6 - 64 = 0$. (*Hint: Begin by factoring the expression as the difference of two perfect squares.*)

99. (6.5) The graph shown displays the variation in daylight from an average of 12 hours per day (i.e., the maximum is 15 hours and the minimum is 9). Use the graph to *approximate* the number of days in a year there are 10.5 or less hours of daylight. Answers may vary.



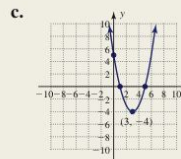
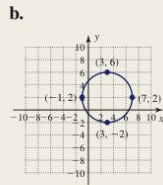
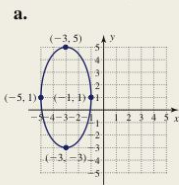
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MID-CHAPTER CHECK

Sketch the graph of each conic section.

1. $(x - 4)^2 + (y + 3)^2 = 9$
2. $x^2 + y^2 - 10x + 4y + 4 = 0$
3. $\frac{(x - 2)^2}{16} + \frac{(y + 3)^2}{1} = 1$
4. $9x^2 + 4y^2 + 18x - 24y + 9 = 0$
5. $\frac{(x + 3)^2}{9} - \frac{(y - 4)^2}{4} = 1$
6. $9x^2 - 4y^2 + 18x - 24y - 63 = 0$
7. Find the equation of each relation and state its domain and range.



8. Solve the following system of inequalities by graphing.

$$\begin{cases} \frac{x^2}{100} + \frac{y^2}{25} \leq 1 \\ x^2 + (y - 4)^2 \leq 36 \end{cases}$$

9. Find the equation of the ellipse (in standard form) if the vertices are $(-4, 0)$ and $(4, 0)$ and the distance between the foci is $4\sqrt{3}$ units.
10. The radio signal emanating from a tall radio tower spreads evenly in all directions with a range of 50 mi. If the tower is located at coordinates $(20, 30)$ and my home is at coordinates $(10, 78)$, will I be able to pick up this station on my home radio? Assume coordinates are in miles.



REINFORCING BASIC CONCEPTS

Ellipses and Hyperbolas with Rational/Irrational Values of a and b

Using the process known as completing the square, we were able to convert from the polynomial form of a conic section to the standard form. However, for some equations, values of a and b are somewhat difficult to identify, since the coefficients are not factors. Consider the equation $20x^2 + 120x + 27y^2 - 54y + 192 = 0$ the equation of an ellipse.

$$\begin{aligned} 20x^2 + 120x + 27y^2 - 54y + 192 &= 0 && \text{original equation} \\ 20(x^2 + 6x + \underline{\quad}) + 27(y^2 - 2y + \underline{\quad}) &= -192 && \text{subtract 192, begin process} \\ 20(x^2 + 6x + 9) + 27(y^2 - 2y + 1) &= -192 + 27 + 180 && \text{complete the square in } x \text{ and } y \\ 20(x + 3)^2 + 27(y - 1)^2 &= 15 && \text{factor and simplify} \\ \frac{4(x + 3)^2}{3} + \frac{9(y - 1)^2}{5} &= 1 && \text{standard form} \end{aligned}$$

Unfortunately, we cannot easily identify the values of a and b , since the coefficients of each binomial square were not "1." In these cases, we can write the equation in standard form by using a simple property of fractions—the numerator and denominator of any fraction can be divided by the same quantity to obtain an equivalent fraction. Although the result may look odd, it can nevertheless be applied here, giving a result of $\frac{(x + 3)^2}{3/4} + \frac{(y - 1)^2}{5/9} = 1$. We can now identify a and b by writing these denominators in squared form,

which gives the following expression: $\frac{(x + 3)^2}{\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{(y - 1)^2}{\left(\frac{\sqrt{5}}{3}\right)^2} = 1$. The values of a and b are now easily seen as

$a \approx 0.866$ and $b \approx 0.745$. Use this idea to complete the following exercises.

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10-47

Exercise 1: Identify the values of a and b by writing the equation $100x^2 - 400x - 18y^2 - 108y + 230 = 0$ in standard form.

Exercise 2: Identify the values of a and b by writing the equation $28x^2 - 56x + 48y^2 + 192y + 195 = 0$ in standard form.

Exercise 3: Write the equation in standard form, then identify the values of a and b and use them to graph the ellipse.

$$\frac{4(x + 3)^2}{49} + \frac{25(y - 1)^2}{36} = 1$$

Section 10.5 Polar Coordinates, Equations, and Graphs

965

Exercise 4: Write the equation in standard form, then identify the values of a and b and use them to graph the hyperbola.

$$\frac{9(x + 3)^2}{80} - \frac{4(y - 1)^2}{81} = 1$$

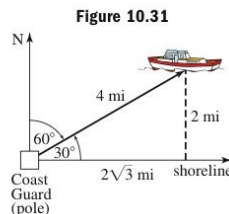
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10.5 Polar Coordinates, Equations, and Graphs

Learning Objectives

In Section 10.5 you will learn how to:

- A. Plot points given in polar form
- B. Convert from rectangular form to polar form
- C. Convert from polar form to rectangular form
- D. Sketch basic polar graphs using an r -value analysis
- E. Use symmetry and families of curves to write a polar equation given a polar graph or information about the graph



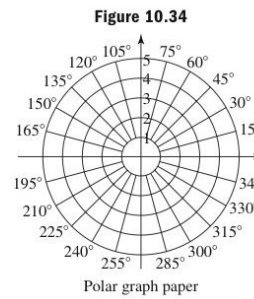
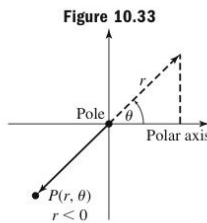
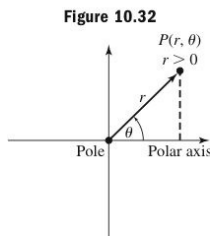
One of the most enduring goals of mathematics is to express relations with the greatest possible simplicity and ease of use. For $\frac{\tan \theta - \cot \theta}{\tan^2 \theta - \cot^2 \theta} = \sin \theta \cos \theta$, we would definitely prefer working with $\sin \theta \cos \theta$, although the expressions are equivalent. Similarly, we would prefer computing $(3 + \sqrt{3}i)^6$ in trigonometric form rather than algebraic form—and would quickly find the result is -1728 . In just this way, many equations and graphs are easier to work with in **polar form** rather than rectangular form. In rectangular form, a circle of radius 2 centered at $(0, 2)$ has the equation $x^2 + (y - 2)^2 = 4$. In polar form, the equation of the same circle is simply $r = 4 \sin \theta$. As you'll see, polar coordinates offer an alternative method for plotting points and graphing relations.

A. Plotting Points Using Polar Coordinates

Suppose a Coast Guard station receives a distress call from a stranded boat. The boater could attempt to give the location in rectangular form, but this might require imposing an arbitrary coordinate grid on an uneven shoreline, using uncertain points of reference. However, if the radio message said, "We're stranded 4 miles out, bearing 60° ," the Coast Guard could immediately locate the boat and send help. In **polar coordinates**, "4 miles out, bearing 60° " would simply be written $(r, \theta) = (4, 30^\circ)$, with r representing the distance from the station and $\theta > 0$ measured from a horizontal axis in the counterclockwise direction as before (see Figure 10.31). If we placed the scenario on a rectangular grid (assuming a straight shoreline), the coordinates of the boat would be $(2\sqrt{3}, 2)$ using basic trigonometry. As you see, the **polar coordinate system** uses angles and distances to locate a point in the plane. In this example, the Coast Guard station would be considered the **pole** or origin, with the x -axis as the **polar axis** or axis of reference (Figure 10.32). A distinctive feature of polar coordinates is that we *allow r to be negative*, in which case $P(r, \theta)$ is the point $|r|$ units from the pole in a direction opposite (180°) to that of θ (Figure 10.33). For convenience, polar graph paper is often used when working with polar coordinates. It consists of a series of concentric circles that share the same center and have integer radii. The standard angles are marked off in multiples of $\frac{\pi}{12} = 15^\circ$ depending on whether you're working in radians or degrees

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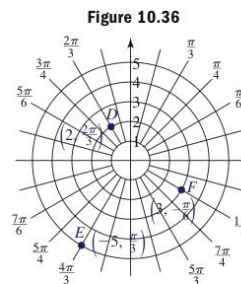
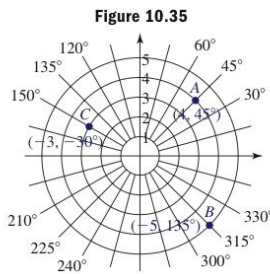
(Figure 10.34). To plot the point $P(r, \theta)$, go a distance of $|r|$ at 0° then move θ° counterclockwise along a circle of radius r . If $r > 0$, plot a point at that location (you're finished). If $r < 0$, the point is plotted on a circle of the same radius, but 180° in the opposite direction.



EXAMPLE 1 ▶ Plotting Points in Polar Coordinates

Plot each point $P(r, \theta)$ given $A(4, 45^\circ)$; $B(-5, 135^\circ)$; $C(-3, -30^\circ)$; $D\left(2, \frac{2\pi}{3}\right)$; $E\left(-5, \frac{\pi}{3}\right)$; and $F\left(3, -\frac{\pi}{6}\right)$.

Solution ▶ For $A(4, 45^\circ)$ go 4 units at 0° , then rotate 45° counterclockwise and plot point A . For $B(-5, 135^\circ)$, move $|-5| = 5$ units at 0° , rotate 135° , then actually plot point B 180° in the opposite direction, as shown. Point $C(-3, -30^\circ)$ is plotted by moving $|-3| = 3$ units at 0° , rotating -30° , then plotting point C 180° in the opposite direction (since $r < 0$). See Figure 10.35. The points $D\left(2, \frac{2\pi}{3}\right)$, $E\left(-5, \frac{\pi}{3}\right)$, and $F\left(3, -\frac{\pi}{6}\right)$ are plotted on the grid in Figure 10.36.



Now try Exercises 7 through 22 ▶

While plotting the points $B(-5, 135^\circ)$ and $F\left(3, -\frac{\pi}{6}\right)$, you likely noticed that the coordinates of a point in polar coordinates are not unique. For $B(-5, 135^\circ)$ it appears more natural to name the location $(5, 315^\circ)$; while for $F\left(3, -\frac{\pi}{6}\right)$, the expression

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Section 10.5 Polar Coordinates, Equations, and Graphs

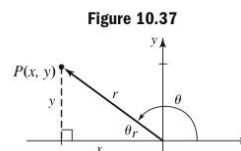
967

A. You've just learned how to plot points given in polar form

$(3, \frac{11\pi}{6})$ is just as reasonable. In fact, for any point $P(r, \theta)$ in polar coordinates, $P(r, \theta \pm 2\pi)$ and $P(-r, \theta \pm \pi)$ name the same location. **See Exercises 23 through 36.**

B. Converting from Rectangular Coordinates to Polar Coordinates

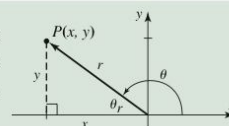
Conversions between rectangular and polar coordinates is a simple application of skills from previous sections, and closely resembles the conversion from the rectangular form to the trigonometric form of a complex number. To make the connection, we first assume $r > 0$ with θ in Quadrant II (see Figure 10.37). In rectangular form, the coordinates of the point are simply (x, y) , with the lengths of x and y forming the sides of a right triangle. The distance r from the origin to point P resembles the modulus of a complex number and is computed in the same way:



$r = \sqrt{x^2 + y^2}$. As long as $x \neq 0$, we have $\theta_r = \tan^{-1}(\frac{y}{x})$, noting θ_r is a reference angle if the terminal side is not in Quadrant I. If needed, refer to Section 5.3 for a review of reference arcs and reference angles.

Converting from Rectangular to Polar Coordinates

Any point $P(x, y)$ in rectangular coordinates can be represented as $P(r, \theta)$ in polar coordinates, where $r = \sqrt{x^2 + y^2}$ and $\theta_r = \tan^{-1}(\frac{y}{x}), x \neq 0$.



EXAMPLE 2 ▶ **Converting a Point from Rectangular Form to Polar Form**

Convert from rectangular to polar form, with $r > 0$ and $0 \leq \theta \leq 360^\circ$ (round values to one decimal place as needed).

- a. $P(-5, 12)$ b. $P(3\sqrt{2}, -3\sqrt{2})$

Solution ▶ a. Point $P(-5, 12)$ is in Quadrant II.

$$\begin{aligned} r &= \sqrt{(-5)^2 + 12^2} & \theta &= \tan^{-1}\left(\frac{12}{-5}\right) \\ &= \sqrt{169} & \theta_r &\approx -67.4^\circ \\ &= 13 & \theta &\approx 112.6^\circ \\ & & P(-5, 12) &\rightarrow P(13, 112.6^\circ) \end{aligned}$$

b. Point $P(3\sqrt{2}, -3\sqrt{2})$ is in Quadrant IV.

$$\begin{aligned} r &= \sqrt{(3\sqrt{2})^2 + (-3\sqrt{2})^2} & \theta &= \tan^{-1}\left(\frac{-3\sqrt{2}}{3\sqrt{2}}\right) \\ &= \sqrt{36} & \theta_r &= -45^\circ \\ &= 6 & \theta &= 315^\circ \\ & & P(3\sqrt{2}, -3\sqrt{2}) &\rightarrow P(6, 315^\circ) \end{aligned}$$

B. You've just learned how to convert from rectangular form to polar form

Now try Exercises 37 through 44 ▶

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C. Converting from Polar Coordinates to Rectangular Coordinates

The conversion from polar form to rectangular form is likewise straightforward. From Figure 10.38 we again note $\cos \theta = \frac{x}{r}$ and $\sin \theta = \frac{y}{r}$, giving $x = r \cos \theta$ and $y = r \sin \theta$. The conversion simply consists of making these substitutions and simplifying.

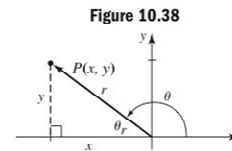
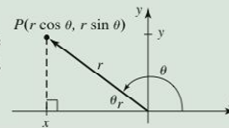


Figure 10.38

Converting from Polar to Rectangular Coordinates

Any point $P(r, \theta)$ in polar coordinates can be represented as $P(x, y)$ in rectangular coordinates, where $x = r \cos \theta$ and $y = r \sin \theta$.



EXAMPLE 3 ▶ Converting a Point from Polar Form to Rectangular Form

Convert from polar to rectangular form (round values to one decimal place as needed).

- a. $P\left(12, \frac{5\pi}{3}\right)$ b. $P(6, 240^\circ)$

Solution ▶ a. Point $P\left(12, \frac{5\pi}{3}\right)$ is in Quadrant IV.

$$\begin{aligned} x &= r \cos \theta & y &= r \sin \theta \\ &= 12 \cos\left(\frac{5\pi}{3}\right) & &= 12 \sin\left(\frac{5\pi}{3}\right) \\ &= 12\left(\frac{1}{2}\right) & &= 12\left(\frac{-\sqrt{3}}{2}\right) \\ &= 6 & &= -6\sqrt{3} \\ P\left(12, \frac{5\pi}{3}\right) &\rightarrow P(6, -6\sqrt{3}) \approx P(6, -10.4) \end{aligned}$$

b. Point $P(6, 240^\circ)$ is in Quadrant III.

$$\begin{aligned} x &= 6 \cos 240^\circ & y &= 6 \sin 240^\circ \\ &= 6\left(-\frac{1}{2}\right) & &= 6\left(\frac{-\sqrt{3}}{2}\right) \\ &= -3 & &= -3\sqrt{3} \\ P(6, 240^\circ) &\rightarrow P(-3, -3\sqrt{3}) \approx P(-3, -5.2) \end{aligned}$$

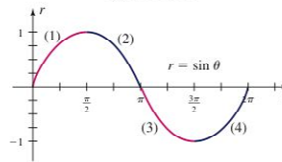
✓ C. You've just learned how to convert from polar form to rectangular form

Now try Exercises 45 through 52 ▶

Using the relationships $x = r \cos \theta$, $y = r \sin \theta$, and $x^2 + y^2 = r^2$, we can actually convert an equation given in polar form, to the equivalent equation in rectangular form. See Exercises 105 and 106.

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Figure 10.40



1. As θ moves from 0 to $\frac{\pi}{2}$, $\sin \theta$ is positive and $|\sin \theta|$ increases from 0 to 1.
 \Rightarrow for $r = \sin \theta$, r is increasing
2. As θ moves from $\frac{\pi}{2}$ to π , $\sin \theta$ is positive and $|\sin \theta|$ decreases from 1 to 0.
 \Rightarrow for $r = \sin \theta$, r is decreasing
3. As θ moves from π to $\frac{3\pi}{2}$, $\sin \theta$ is negative and $|\sin \theta|$ increases from 0 to 1.
 \Rightarrow for $r = \sin \theta$, r is increasing
4. As θ moves from $\frac{3\pi}{2}$ to 2π , $\sin \theta$ is negative and $|\sin \theta|$ decreases from 1 to 0.
 \Rightarrow for $r = \sin \theta$, r is decreasing

In summary, note that the value of $|r|$ goes through four cycles, two where it is increasing from 0 to 1 (in red), and two where it is decreasing from 1 to 0 (in blue).

WORTHY OF NOTE
 It is important to remember that if $r < 0$, the related point on the graph is $|r|$ units from center, 180° in the opposite direction: $(-r, \theta) \rightarrow (r, \theta + 180^\circ)$. In addition, students are encouraged not to use a table of values, a conversion to rectangular coordinates, or a graphing calculator until after the r -value analysis.

EXAMPLE 5 ▶ Graphing Polar Equations Using an r -Value Analysis

Sketch the graph of $r = 4 \sin \theta$ using an r -value analysis.

Solution ▶ Begin by noting that $r = 0$ at $\theta = 0$, and will increase from 0 to 4 as the clock “ticks” from 0 to $\frac{\pi}{2}$, since $\sin \theta$ is increasing from 0 to 1. (1) For $\theta = \frac{\pi}{6}, \frac{\pi}{4}$, and $\frac{\pi}{3}$, $r = 2, r \approx 2.8$, and $r \approx 3.5$, respectively (at $\theta = \frac{\pi}{2}, r = 4$). See Figure 10.41. (2) As θ continues “ticking” from $\frac{\pi}{2}$ to π , $|r|$ decreases from 4 to 0, since $\sin \theta$ is decreasing from 1 to 0. For $\theta = \frac{2\pi}{3}, \frac{3\pi}{4}$, and $\frac{5\pi}{6}$, $r \approx 3.5, r \approx 2.8$, and $r = 2$, respectively (at $\theta = \pi, r = 0$). See Figure 10.42. (3) From π to $\frac{3\pi}{2}$, $|r|$ increases from 0 to 4, but since $r < 0$, this portion of the graph is reflected back into Quadrant I, overlapping the portion already drawn from 0 to $\frac{\pi}{2}$. (4) From $\frac{3\pi}{2}$ to 2π , $|r|$ decreases from 4 to 0, overlapping the portion drawn from $\frac{\pi}{2}$ to π . We conclude the graph is a closed figure limited to Quadrants I and II as shown in Figure 10.42. This is a circle with radius 2, centered at $(0, 2)$. In summary:

$$r = 4 \sin \theta$$

θ	0 to $\frac{\pi}{2}$	$\frac{\pi}{2}$ to π	π to $\frac{3\pi}{2}$	$\frac{3\pi}{2}$ to 2π
$ r $	0 to 4	4 to 0	0 to 4	4 to 0

Now try Exercises 57 and 58 ▶

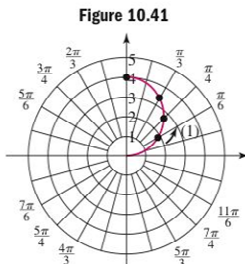
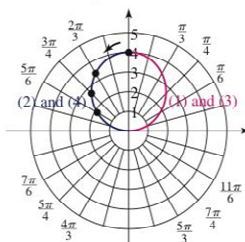


Figure 10.42



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Although it takes some effort, r -value analysis offers an efficient way to graph polar equations, and gives a better understanding of graphing in polar coordinates. In addition, it often enables you to sketch the graph with a minimum number of calculations and plotted points. As you continue using the technique, it will help to have Figure 10.40 in plain view for quick reference, as well as the corresponding analysis of $y = \cos \theta$ for polar graphs involving cosine (see Exercise 98).

EXAMPLE 6 ▶ Graphing Polar Equations Using an r -Value Analysis

Sketch the graph of $r = 2 + 2 \sin \theta$ using an r -value analysis.

Solution ▶ Since the minimum value of $\sin \theta$ is -1 , we note that r will always be greater than or equal to zero. At $\theta = 0$, r has a value of 2 ($\sin 0 = 0$), and will increase from 2 to 4 as the clock “ticks” from 0 to $\frac{\pi}{2}$ ($\sin \theta$ is positive and $|\sin \theta|$ is increasing).

From $\frac{\pi}{2}$ to π , r decreases from 4 to 2 ($\sin \theta$ is positive and $|\sin \theta|$ is decreasing).

From π to $\frac{3\pi}{2}$, r decreases from 2 to 0 ($\sin \theta$ is negative and $|\sin \theta|$ is increasing);

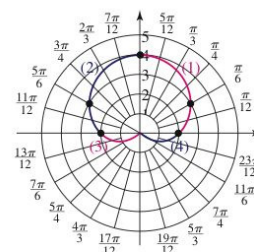
and from $\frac{3\pi}{2}$ to 2π , r increases from 0 to 2 ($\sin \theta$ is negative and $|\sin \theta|$ is decreasing). We conclude the graph is a closed figure containing the points $(2, 0)$, $(4, \frac{\pi}{2})$, $(2, \pi)$, and $(0, \frac{3\pi}{2})$. Noting that $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$ will produce

integer values, we evaluate $r = 2 + 2 \sin \theta$ and obtain the additional points $(3, \frac{\pi}{6})$ and $(3, \frac{5\pi}{6})$. Using these points and the r -value analysis produces the graph shown here, called a **cardioid** (from the limaçon family of curves). In summary we have:

WORTHY OF NOTE
While the same graph is obtained by simply plotting points, using an r -value analysis is often more efficient, particularly with more complex equations.

θ	$r = 4 \sin \theta$
0	0
30	2
45	$2\sqrt{2} \approx 2.8$
60	$2\sqrt{3} \approx 3.5$
90	4
120	$2\sqrt{3} \approx 3.5$
135	$2\sqrt{2} \approx 2.8$
150	2
180	0

θ	$r = 2 + 2 \sin \theta$
(1) 0 to $\frac{\pi}{2}$	2 to 4
(2) $\frac{\pi}{2}$ to π	4 to 2
(3) π to $\frac{3\pi}{2}$	2 to 0
(4) $\frac{3\pi}{2}$ to 2π	0 to 2



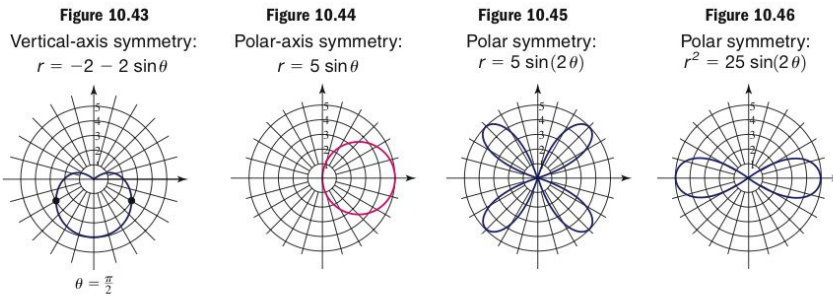
D. You've just learned how to sketch basic polar graphs using an r -value analysis

Now try Exercises 59 through 62 ▶

E. Symmetry and Families of Polar Graphs

Even with a careful r -value analysis, some polar graphs require a good deal of effort to produce. In many cases, symmetry can be a big help, as can recognizing certain families of equations and their related graphs. As with other forms of graphing, gathering this information beforehand will enable you to graph relations with a smaller number of plotted points. Figures 10.43 to 10.46 offer some examples of symmetry for polar graphs.

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WORTHY OF NOTE

In mathematics we refer to the tests for polar symmetry as *sufficient but not necessary conditions*. The tests are *sufficient* to show symmetry (if the test is satisfied, the graph must be symmetric), but the tests are *not necessary* to show symmetry (the graph may be symmetric even if the test is not satisfied).

The tests for symmetry in polar coordinates bear a strong resemblance to those for rectangular coordinates, but there is a major difference. Since there are many different ways to name a point in polar coordinates, a polar graph may actually exhibit a form of symmetry without satisfying the related test. In other words, the tests are *sufficient* to establish symmetry, but not *necessary*.

The formal tests for symmetry are explored in **Exercises 100 to 102**. For our purposes, we'll rely on a somewhat narrower view, one that is actually a synthesis of our observations here and our previous experience with the sine and cosine.

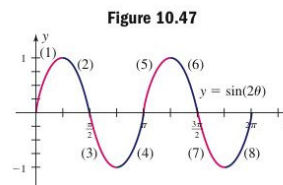
Symmetry for Graphs of Certain Polar Equations

Given the polar equation $r = f(\theta)$,

1. If $f(\theta)$ represents an expression in terms of sine(s), the graph will be symmetric to $\theta = \frac{\pi}{2}$: (r, θ) and $(r, \pi - \theta)$ are on the graph.
2. If $f(\theta)$ represents an expression in terms of cosine(s), the graph will be symmetric to $\theta = 0$: (r, θ) and $(r, -\theta)$ are on the graph.

While the fundamental ideas from Examples 5 and 6 go a long way toward graphing other polar equations, our discussion would not be complete without a review of the *period* of sine and cosine. Many polar equations have factors of $\sin(n\theta)$ or $\cos(n\theta)$ in them, and it helps to recall the period formula $P = \frac{2\pi}{n}$. Comparing $r = 4 \sin \theta$ from

Example 5 with $r = 4 \sin(2\theta)$, we note the period of sine changes from $P = 2\pi$ to $P = \frac{2\pi}{2} = \pi$, meaning there will be twice as many cycles and $|r|$ will now go through *eight* cycles—four where $|\sin(2\theta)|$ is increasing from 0 to 1 (in red), and four where it is decreasing from 1 to 0 (in blue). See Figure 10.47.



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EXAMPLE 7 ▶ Sketching Polar Graphs Using Symmetry and r -Values

Sketch the graph of $r = 4 \sin(2\theta)$ using symmetry and an r -value analysis.

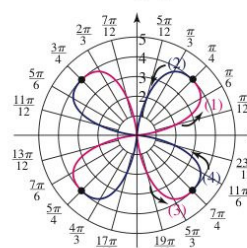
Solution ▶ Since r is expressed in terms of sine, the graph will be symmetric to $\theta = \frac{\pi}{2}$. We note that $r = 0$ at $\theta = \frac{n\pi}{2}$, where n is even, and the graph will go through the pole at these points. This also tells us the graph will be a closed figure. From the graph of $\sin(2\theta)$ in Figure 10.47, we see $|\sin(2\theta)| = 1$ at $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4},$ and $\frac{7\pi}{4}$, so the graph will include the points $(4, \frac{\pi}{4}), (4, \frac{3\pi}{4}), (4, \frac{5\pi}{4}),$ and $(4, \frac{7\pi}{4})$. Only the analysis of the first four cycles is given next, since the remainder of the graph can be drawn using symmetry.

	Cycle	r-Value Analysis	Location of Graph
(1)	0 to $\frac{\pi}{4}$	$ r $ increases from 0 to 4	QI ($r > 0$)
(2)	$\frac{\pi}{4}$ to $\frac{\pi}{2}$	$ r $ decreases from 4 to 0	QI ($r > 0$)
(3)	$\frac{\pi}{2}$ to $\frac{3\pi}{4}$	$ r $ increases from 0 to 4	QIV ($r < 0$)
(4)	$\frac{3\pi}{4}$ to π	$ r $ decreases from 4 to 0	QIV ($r < 0$)

Plotting the points and applying the r -value analysis with the symmetry involved produces the graph in the figure, called a **four-leaf rose**. At any time during this process, additional points can be calculated to “round-out” the graph.

$r = 4 \sin(2\theta)$

θ	to	r-Value Analysis
0	to $\frac{\pi}{4}$	0 to 4
$\frac{\pi}{4}$	to $\frac{\pi}{2}$	4 to 0
$\frac{\pi}{2}$	to $\frac{3\pi}{4}$	0 to 4
$\frac{3\pi}{4}$	to π	4 to 0



Now try Exercises 63 through 70 ▶

Graphing Polar Equations

To assist the process of graphing polar equations:

1. Carefully note any symmetries you can use.
2. Have graphs of $y = \sin(n\theta)$ and $y = \cos(n\theta)$ in view for quick reference.
3. Use these graphs to analyze the value of r as the “clock ticks” around the polar grid: (a) determine the max/min r -values and write them in polar form, and (b) determine the polar-axis intercepts and write them in polar form.
4. Plot the points, then use the r -value analysis and any symmetries to complete the graph.

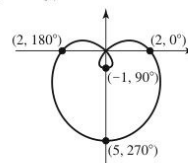
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Similar to polynomial graphs, polar graphs come in numerous shapes and varieties, yet many of them share common characteristics and can be organized into certain families. Some of the more common families are illustrated in Appendix V, and give the general equation and related graph for common family members. Also included are characteristics of certain graphs that will enable you to develop the polar equation given its graph or information about its graph. For further investigations using a graphing calculator, see Exercises 71 through 76.

EXAMPLE 8 ▶ Graphing a Limaçon Using Stated Conditions

Find the equation of the polar curve satisfying the given conditions, then sketch the graph: limaçon, symmetric to $\theta = 90^\circ$, with $a = 2$ and $b = -3$.

Solution ▶ The general equation of a limaçon symmetric to $\theta = 90^\circ$ is $r = a + b \sin \theta$, so our desired equation is $r = 2 - 3 \sin \theta$. Since $|a| < |b|$, the limaçon has an inner loop of length $3 - 2 = 1$ and a maximum distance from the origin of $2 + 3 = 5$. The polar-axis intercepts are $(2, 0)$ and $(2, 180^\circ)$. With $b < 0$, the graph is reflected across the polar axis (facing “downward”). The complete graph is shown in the figure.

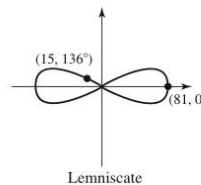


Now try Exercises 79 through 94 ▶

EXAMPLE 9 ▶ Modeling the Flight Path of a Scavenger Bird

Scavenger birds sometimes fly over dead or dying animals (called carrion) in a “figure-eight” formation, closely resembling the graph of a lemniscate. Suppose the flight path of one of these birds was plotted and found to contain the polar coordinates $(81, 0^\circ)$ and $(0, 45^\circ)$. Find the equation of the lemniscate. If the bird lands at the point $(r, 136^\circ)$, how far is it from the carrion? Assume r is in yards.

Solution ▶ Since $(81, 0^\circ)$ is a point on the graph, the lemniscate is symmetric to the polar axis and the general equation is $r^2 = a^2 \cos(2\theta)$. The point $(81, 0^\circ)$ indicates $a = 81$, hence the equation is $r^2 = 6561 \cos(2\theta)$. At $\theta = 136^\circ$ we have $r^2 = 6561 \cos 272^\circ$, and the bird has landed $r \approx 15$ yd away.



Now try Exercises 95 through 97 ▶

✓ **E.** You've just learned how to use symmetry and families of curves to write a polar equation given a polar graph or information about the graph

You've likely been wondering how the different families of polar graphs were named. The roses are easy to figure as each graph has a flower-like appearance. The limaçon (pronounced li-ma-sawn) family takes its name from the Latin words *limax* or *lamacis*, meaning “snail.” With some imagination, these graphs do have the appearance of a snail shell. The cardioids are a subset of the limaçon family and are so named due to their obvious resemblance to the human heart. In fact, the name stems from the Greek *kardia* meaning heart, and many derivative words are still in common use (a cardiologist is one who specializes in a study of the heart). Finally, there is the lemniscate family, a name derived from the Latin *lemniscus*, which describes a certain kind of ribbon. Once again, a little creativity enables us to make the connection between ribbons, bows, and the shape of this graph.

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10.5 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The point (r, θ) is said to be written in _____ coordinates.
2. In polar coordinates, the origin is called the _____ and the horizontal axis is called the _____ axis.
3. The point $(4, 135^\circ)$ is located in Q _____, while $(-4, 135^\circ)$ is located in Q _____.

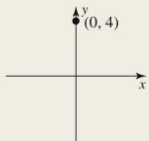
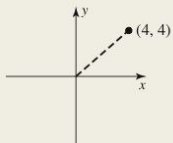
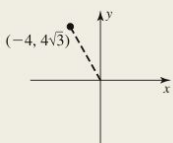

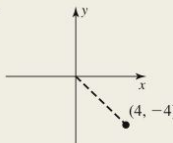
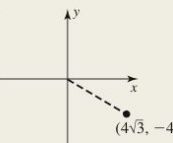
4. If a polar equation is given in terms of cosine, the graph will be symmetric to _____.
5. Write out the procedure for plotting points in polar coordinates, as though you were explaining the process to a friend.
6. Discuss the graph of $r = 6 \cos \theta$ in terms of an r -value analysis, using $y = \cos \theta$ and a color-coded graph.

► DEVELOPING YOUR SKILLS

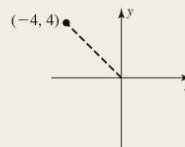
Plot the following points using polar graph paper.

- | | | |
|-----------------------------|----------------------------|----------------------------|
| 7. $(4, \frac{\pi}{2})$ | 8. $(3, \frac{3\pi}{2})$ | 9. $(2, \frac{5\pi}{4})$ |
| 10. $(4.5, -\frac{\pi}{3})$ | 11. $(-5, \frac{5\pi}{6})$ | 12. $(-4, \frac{7\pi}{4})$ |
| 13. $(-3, -\frac{2\pi}{3})$ | 14. $(-4, -\frac{\pi}{4})$ | |

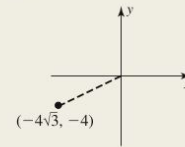
Express the points shown using polar coordinates with θ in radians, $0 \leq \theta < 2\pi$ and $r > 0$.

- | | |
|--|--|
| <p>15. </p> <p>17. </p> <p>19. </p> | <p>16. </p> <p>18. </p> <p>20. </p> |
|--|--|

21.



22.

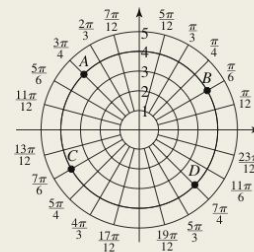


List three alternative ways the given points can be expressed in polar coordinates using $r > 0$, $r < 0$, and $\theta \in [-2\pi, 2\pi)$.

- | | |
|-----------------------------------|------------------------------------|
| 23. $(3\sqrt{2}, \frac{3\pi}{4})$ | 24. $(4\sqrt{3}, -\frac{5\pi}{3})$ |
| 25. $(-2, \frac{11\pi}{6})$ | 26. $(-3, -\frac{7\pi}{6})$ |

Match each (r, θ) given to one of the points A, B, C, or D shown.

Exercise 27-36



- | | |
|----------------------------|----------------------------|
| 27. $(4, -\frac{5\pi}{6})$ | 28. $(4, -\frac{5\pi}{4})$ |
|----------------------------|----------------------------|

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29. $\left(-4, \frac{\pi}{6}\right)$ 30. $\left(-4, \frac{3\pi}{4}\right)$
 31. $\left(-4, -\frac{5\pi}{4}\right)$ 32. $\left(-4, -\frac{\pi}{4}\right)$
 33. $\left(4, \frac{13\pi}{6}\right)$ 34. $\left(4, \frac{19\pi}{6}\right)$
 35. $\left(-4, -\frac{21\pi}{4}\right)$ 36. $\left(4, -\frac{35\pi}{6}\right)$

Convert from rectangular coordinates to polar coordinates. A diagram may help.

37. $(-8, 0)$ 38. $(0, -7)$
 39. $(4, 4)$ 40. $(4\sqrt{3}, 4)$
 41. $(5\sqrt{2}, 5\sqrt{2})$ 42. $(6, -6\sqrt{3})$
 43. $(-5, -12)$ 44. $(-3.5, 12)$

Convert from polar coordinates to rectangular coordinates. A diagram may help.

45. $(8, 45^\circ)$ 46. $(6, 60^\circ)$
 47. $\left(4, \frac{3\pi}{4}\right)$ 48. $\left(5, \frac{5\pi}{6}\right)$
 49. $\left(-2, \frac{7\pi}{6}\right)$ 50. $\left(-10, \frac{4\pi}{3}\right)$
 51. $(-5, -135^\circ)$ 52. $(-4, -30^\circ)$

▶ WORKING WITH FORMULAS

77. The midpoint formula in polar coordinates:

$$M = \left(\frac{r \cos \alpha + R \cos \beta}{2}, \frac{r \sin \alpha + R \sin \beta}{2} \right)$$

The midpoint of a line segment connecting the points (r, α) and (R, β) in polar coordinates can be found using the formula shown. Find the midpoint of the line segment between $(r, \alpha) = (6, 45^\circ)$ and $(R, \beta) = (8, 30^\circ)$, then convert these points to rectangular coordinates and find the midpoint using the "standard" formula. Do the results match?


▶ APPLICATIONS

Polar graphs: Find the equation of a polar graph satisfying the given conditions, then sketch the graph.

79. limaçon, symmetric to polar axis, $a = 4$ and $b = 4$

Sketch each polar graph using an r -value analysis (a table may help), symmetry, and any convenient points.

53. $r = 5$ 54. $r = 6$
 55. $\theta = \frac{\pi}{6}$ 56. $\theta = -\frac{3\pi}{4}$
 57. $r = 4 \cos \theta$ 58. $r = 2 \sin \theta$
 59. $r = 3 + 3 \sin \theta$ 60. $r = 2 + 2 \cos \theta$
 61. $r = 2 - 4 \sin \theta$ 62. $r = 1 - 2 \cos \theta$
 63. $r = 5 \cos(2\theta)$ 64. $r = 3 \sin(4\theta)$
 65. $r = 4 \sin 2\theta$ 66. $r = 6 \cos(5\theta)$
 67. $r^2 = 9 \sin(2\theta)$ 68. $r^2 = 16 \cos(2\theta)$
 69. $r = 4 \sin\left(\frac{\theta}{2}\right)$ 70. $r = 6 \cos\left(\frac{\theta}{2}\right)$

 Use a graphing calculator in polar mode to produce the following polar graphs.

71. $r = 4\sqrt{1 - \sin^2\theta}$, a hippopede
 72. $r = 3 + \csc \theta$, a conchoid
 73. $r = 2 \cos \theta \cot \theta$, a cissoid
 74. $r = \cot \theta$, a kappa curve
 75. $r = 8 \sin \theta \cos^2\theta$, a bifoliate
 76. $r = 8 \cos \theta(4 \sin^2\theta - 2)$, a folium

78. The distance formula in polar coordinates:

$$d = \sqrt{R^2 + r^2 - 2Rr \cos(\alpha - \beta)}$$

Using the law of cosines, it can be shown that the distance between the points (R, α) and (r, β) in polar coordinates is given by the formula indicated. Use the formula to find the distance between $(R, \alpha) = (6, 45^\circ)$ and $(r, \beta) = (8, 30^\circ)$, then convert these to rectangular coordinates and compute the distance between them using the "standard" formula. Do the results match?

80. rose, four petals, two petals symmetric to the polar axis, $a = 6$

81. rose, five petals, one petal symmetric to the polar axis, $a = 4$

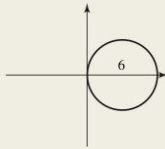
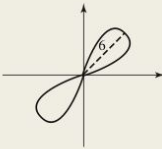
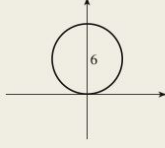
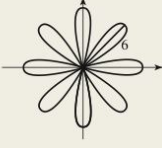
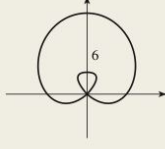
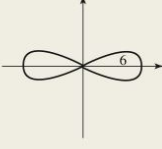
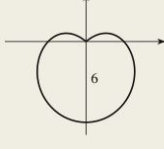
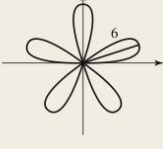
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10-59

Section 10.5 Polar Coordinates, Equations, and Graphs

- 82. limaçon, symmetric to $\theta = \frac{\pi}{2}$, $a = 2$ and $b = 4$
- 83. lemniscate, $a = 4$ through $(\pi, 4)$
- 84. lemniscate, $a = 8$ through $(8, \frac{\pi}{4})$
- 85. circle, symmetric to $\theta = \frac{\pi}{2}$, center at $(2, \frac{\pi}{2})$, containing $(2, \frac{\pi}{6})$
- 86. circle, symmetric to polar axis, through $(6, \pi)$

Matching: Match each graph to its equation a through h, which follow. Justify your answers.

87.		88.	
89.		90.	
91.		92.	
93.		94.	

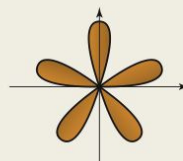
- a. $r = 6 \cos \theta$
- b. $r = 3 - 3 \sin \theta$
- c. $r = 6 \cos(4\theta)$
- d. $r^2 = 36 \cos(2\theta)$
- e. $r^2 = 36 \sin(2\theta)$
- f. $r = 2 + 4 \sin \theta$
- g. $r = 6 \sin \theta$
- h. $r = 6 \sin(5\theta)$

95. **Figure eights:** Waiting for help to arrive on foot, a light plane is circling over some stranded hikers using a “figure eight” formation, closely resembling the graph of a lemniscate. Suppose the flight path of the plane was plotted (using the hikers as the origin) and found to contain the polar coordinates $(7200, 45^\circ)$ and $(0, 90^\circ)$ with r in meters. Find the equation of the lemniscate.



96. **Animal territories:** Territorial animals often prowl the borders of their territory, marking the boundaries with various bodily excretions. Suppose the territory of one such animal was limaçon shaped, with the pole representing the den of the animal. Find the polar equation defining the animal's territory if markings are left at $(750, 0^\circ)$, $(1000, 90^\circ)$, and $(750, 180^\circ)$. Assume r is in meters.

97. **Prop manufacturing:** The propellers for a toy boat are manufactured by stamping out a rose with n petals and then bending each blade. If the manufacturer wants propellers with five blades and a radius of 15 mm, what two polar equations will satisfy these specifications?



98. **Polar curves and cosine:** Do a complete r -value analysis for graphing polar curves involving cosine. Include a color-coded graph showing the relationship between r and θ , similar to the analysis for sines that preceded Example 6.

► EXTENDING THE CONCEPT

99. The polar graph $r = a\theta$ is called the *Spiral of Archimedes*. Consider the spiral $r = \frac{1}{2}\theta$. As this graph spirals around the origin, what is the distance between each positive, polar intercept? In QI, what is the distance between consecutive branches of the

spiral each time it intersects $\theta = \frac{\pi}{4}$? What is the distance between consecutive branches of the spiral at $\theta = \frac{\pi}{2}$? What can you conclude?

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As mentioned in the exposition, tests for symmetry of polar graphs are sufficient to show symmetry (if the test is satisfied, the graph must be symmetric), but the tests are not necessary to show symmetry (the graph may be symmetric even if the test is not satisfied). For $r = f(\theta)$, the formal tests for the symmetry are: (1) the graph will be symmetric to the polar axis if $f(\theta) = f(-\theta)$; (2) the graph will be symmetric to the line $\theta = \frac{\pi}{2}$ if

$f(\pi - \theta) = f(\theta)$; and (3) the graph will be symmetric to the pole if $f(\theta) = -f(\theta)$.

100. Sketch the graph of $r = 4 \sin(2\theta)$. Show the equation fails the first test, yet the graph is still symmetric to the polar axis.
101. Why is the graph of every lemniscate symmetric to the pole?
102. Verify that the graph of every limaçon of the form $r = a + b \cos \theta$ is symmetric to the polar axis.

► MAINTAINING YOUR SKILLS

106. (6.2) Verify the following is an identity:
 $\cos^2 x - \sin^2 x = 1 - \sin(2x) \tan x$.

107. (6.7) Solve for $t \in [0, 2\pi)$:
 $20 = 5 - 30 \sin\left(2t - \frac{\pi}{6}\right)$.

108. (1.3) Solve the absolute value inequality. Answer in interval notation:
 $-3|2x + 5| - 7 > -19$

103. The graphs of $r = a \sin(n\theta)$ and $r = a \cos(n\theta)$ are from the rose family of polar graphs. If n is odd, there are n petals in the rose, and if n is even, there are $2n$ petals. An interesting extension of this fact is that the n petals enclose exactly 25% of the area of the circumscribed circle, and the $2n$ petals enclose exactly 50%. Find the area within the boundaries of the rose defined by $r = 6 \sin(5\theta)$.

To develop an understanding of polar equations, we used the following facts $x^2 + y^2 = r^2$, $x = r \cos \theta$, and $y = r \sin \theta$. Using these relationships, we can actually convert polar equations to rectangular equations and vice versa, showing that a particular equation *can be graphed in either form*. Use these relationships to write these polar equations in rectangular form. (*Hint: Isolate the term kr (k a constant) on one side, then square.*)

104. $r = \frac{1}{1 + \sin \theta}$ 105. $r = \frac{6}{2 + 4 \sin \theta}$

109. (2.7) Graph the piecewise function shown and state its domain and range.

$$f(x) = \begin{cases} x + 2 & -5 \leq x < -1 \\ x & -1 \leq x < 2 \\ 4 & 2 < x \leq 5 \end{cases}$$

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10.6 More on the Conic Sections: Rotation of Axes and Polar Form

Learning Objectives

In Section 10.6 you will learn how to:

- A.** Graph conic sections that have nonvertical and nonhorizontal axes (rotated conics)
- B.** Identify conics using the discriminant of the polynomial form—the invariant $B^2 - 4AC$
- C.** Write the equation of a conic section in polar form
- D.** Solve applications involving the conic sections in polar form

Our study of conic sections would not be complete without considering conic sections whose graphs are not symmetric to a vertical or horizontal axis. The axis of symmetry still exists, but is rotated by some angle. We'll first study these **rotated conics** using the equation in its polynomial form, then investigate some interesting applications of the polar form.

A. Rotated Conics and the Rotation of Axes

It's always easier to understand a new idea in terms of a known idea, so we begin our study with a review of the reciprocal function $y = \frac{1}{x}$. From the equation we note:

1. The denominator is zero when $x = 0$, and the y -axis is a vertical asymptote (the vertical line $x = 0$).
2. Since the degree of the numerator is less than the degree of the denominator, the x -axis is a horizontal asymptote (the horizontal line $y = 0$).
3. Since $x < 0$ implies $y < 0$ and $x > 0$ implies $y > 0$, the graph will have two branches—one in the first quadrant and one in the third.

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Note the polynomial form of this equation is $xy = 1$. The resulting graph is shown in Figure 10.48, and is actually the graph of a hyperbola with a transverse axis of $y = x$. Using the 45-45-90 triangle indicated, we find the distance from the origin to each vertex is $\sqrt{2}$. If we rotated the hyperbola 45° clockwise, we would obtain a more "standard" graph with a horizontal transverse axis and vertices at $(\pm a, 0) \rightarrow (\pm\sqrt{2}, 0)$. The asymptotes would be $y = \pm 1x$, and since $y = \pm \frac{b}{a}x$ is the general form we know $b = \pm\sqrt{2}$. This information can be used to find the equation of the rotated hyperbola.

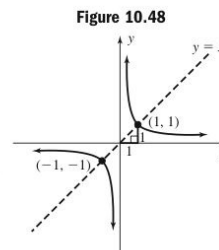
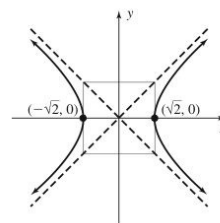


Figure 10.48

EXAMPLE 1 ▶ Finding the Equation of a Rotated Conic from Its Graph

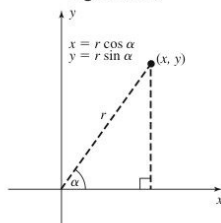
The hyperbola $xy = 1$ is rotated clockwise 45°, with new vertices at $(\pm\sqrt{2}, 0)$, asymptotes at $y = \pm 1x$ and $b = \pm\sqrt{2}$. Find the equation and graph the hyperbola.

Solution ▶ Using the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and substituting $\pm\sqrt{2}$ for a and b , the equation of the rotated hyperbola is $\frac{x^2}{2} - \frac{y^2}{2} = 1$ or $x^2 - y^2 = 2$ in polynomial form. The resulting graph is the central hyperbola shown.



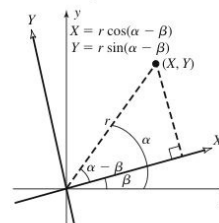
Now try Exercises 7 and 8 ▶

Figure 10.49



It's important to note the equation of the rotated hyperbola is devoid of the mixed "xy" term. In nondegenerate cases, the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is the polynomial form of a conic with axes that are vertical/horizontal. However, the most general form of the equation is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, and includes this Bxy term. As noted in Example 1, the inclusion of this term will rotate the graph through some angle β . Based on these observations, we reason that one approach to graphing these conics is to find the angle of rotation β with respect to the xy -axes. We can then use β to rewrite the equation so that it corresponds to a new set of XY -axes, which are parallel to the axes of the conic. The mixed xy -term will be absent from the new equation and we can graph the conic on the new axes using the same ideas as before (identifying a, b , foci, and so on). To find β , recall that a point (x, y) in the xy -plane can be written $x = r \cos \alpha, y = r \sin \alpha$, as in Figure 10.49. The diagram in Figure 10.50 shows the axes of a new XY -plane, rotated counterclockwise by angle β . In this new plane, the coordinates of the point (x, y) become $X = r \cos(\alpha - \beta)$ and $Y = r \sin(\alpha - \beta)$ as shown. Using the difference identities for sine and cosine and substituting $x = r \cos \alpha$ and $y = r \sin \alpha$ leads to

Figure 10.50



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$$\begin{aligned}
 X &= r \cos(\alpha - \beta) & Y &= r \sin(\alpha - \beta) \\
 &= r(\cos \alpha \cos \beta + \sin \alpha \sin \beta) & &= r(\sin \alpha \cos \beta - \cos \alpha \sin \beta) \\
 &= r \cos \alpha \cos \beta + r \sin \alpha \sin \beta & &= r \sin \alpha \cos \beta - r \cos \alpha \sin \beta \\
 &= x \cos \beta + y \sin \beta & &= y \cos \beta - x \sin \beta
 \end{aligned}$$

The last two equations can be written as a system, which we will use to solve for x and y in terms of X and Y .

$$\begin{cases}
 X = x \cos \beta + y \sin \beta & \text{original system} \\
 Y = y \cos \beta - x \sin \beta &
 \end{cases}$$

$$\begin{cases}
 X \cos \beta = x \cos^2 \beta + y \sin \beta \cos \beta & \text{multiply first equation by } \cos \beta \\
 Y \sin \beta = y \sin \beta \cos \beta - x \sin^2 \beta & \text{multiply second equation by } \sin \beta
 \end{cases}$$

$$\begin{aligned}
 X \cos \beta - Y \sin \beta &= x \cos^2 \beta + x \sin^2 \beta & \text{first equation} - \text{second equation} \\
 X \cos \beta - Y \sin \beta &= x & \text{factor out } x(\cos^2 \beta + \sin^2 \beta = 1)
 \end{aligned}$$

WORTHY OF NOTE

If you are familiar with matrices, it may be easier to remember the rotation formulas in their matrix form, since the pattern of functions is the same, with only a difference in sign:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

See Exercises 86 and 87.

Re-solving the system for y results in $y = X \sin \beta + Y \cos \beta$, yielding what are called the **rotation of axes formulas** (see Exercise 79).

Rotation of Axes Formulas

If the x - and y -axes of the xy -plane are rotated counterclockwise by the (acute) angle β to form the X - and Y -axes of an XY -plane, the coordinates of the points (x, y) and (X, Y) are related by the formulas

$$\begin{aligned}
 x &= X \cos \beta - Y \sin \beta & X &= x \cos \beta + y \sin \beta \\
 y &= X \sin \beta + Y \cos \beta & Y &= -x \sin \beta + y \cos \beta
 \end{aligned}$$

EXAMPLE 2 ▶ Naming the Location of a Point After Rotating the Axes

Given the point $(1, \sqrt{3})$ in the xy -plane, find the coordinates of this point in the XY -plane given the angle β between the xy -axes and the XY -axes is 60° .

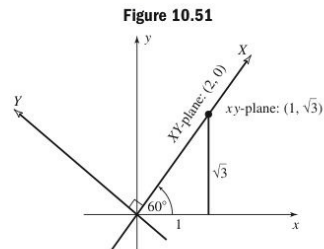
Solution ▶ Using the formulas with $x = 1$, $y = \sqrt{3}$, and $\beta = 60^\circ$, we obtain

$$\begin{aligned}
 X &= x \cos \beta + y \sin \beta & Y &= -x \sin \beta + y \cos \beta \\
 &= 1 \cos 60^\circ + \sqrt{3} \sin 60^\circ & &= -1 \sin 60^\circ + \sqrt{3} \cos 60^\circ \\
 &= \left(\frac{1}{2}\right) + \sqrt{3} \left(\frac{\sqrt{3}}{2}\right) & &= -\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \\
 &= 2 & &= 0
 \end{aligned}$$

The coordinates of $P(X, Y)$ would be $(2, 0)$.

Now try Exercises 9 through 16 ▶

The diagram in Figure 10.51 provides a more intuitive look at the rotation from Example 2. As you can see, a 30-60-90 triangle is formed with a hypotenuse of 2, giving coordinates $(2, 0)$ in the XY -plane.



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EXAMPLE 3 ▶ Writing the Equation of a Conic After Rotating the Axes

The ellipse $X^2 + 4Y^2 = 16$ is rotated clockwise 45° . What is the corresponding equation in the xy -plane?

Solution ▶ We proceed as before, using the rotation formulas $X = x \cos \beta + y \sin \beta$ and $Y = y \cos \beta - x \sin \beta$. With $\beta = 45^\circ$ we have $\cos \beta = \sin \beta = \frac{\sqrt{2}}{2}$, yielding

$$\begin{aligned}
 X^2 + 4Y^2 &= 16 \\
 (x \cos \beta + y \sin \beta)^2 + 4(y \cos \beta - x \sin \beta)^2 &= 16 && \text{use rotation formulas} \\
 \left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2 + 4\left(\frac{\sqrt{2}}{2}y - \frac{\sqrt{2}}{2}x\right)^2 &= 16 && \text{substitute } \frac{\sqrt{2}}{2} \text{ for } \sin \beta \text{ and } \cos \beta \\
 \left(\frac{1}{2}x^2 + xy + \frac{1}{2}y^2\right) + 4\left(\frac{1}{2}x^2 - xy + \frac{1}{2}y^2\right) &= 16 && \text{square binomials} \\
 \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + 2x^2 - 4xy + 2y^2 &= 16 && \text{distribute} \\
 \frac{5}{2}x^2 - 3xy + \frac{5}{2}y^2 &= 16 && \text{result}
 \end{aligned}$$

Now try Exercises 17 through 20 ▶

Note the equation of the conic in the standard xy -plane contains the “mixed” Bxy -term. In practice, we seek to reverse this procedure by starting in the xy -plane, and finding the angle β needed to *eliminate* the Bxy -term. Using the rotation formulas and the appropriate angle β , the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ becomes $aX^2 + cY^2 + dX + eY + f = 0$, where the xy -term is absent. To find the angle β , note that without loss of generality, we can assume $D = E = 0$ since only the second-degree terms are used to identify a conic. Starting with the simplified equation $Ax^2 + Bxy + Cy^2 + F = 0$ and using the rotation formulas we obtain

$$\begin{aligned}
 Ax^2 + Bxy + Cy^2 + F &= 0 \\
 A(X \cos \beta - Y \sin \beta)^2 + B(X \cos \beta - Y \sin \beta)(X \sin \beta + Y \cos \beta) + C(X \sin \beta + Y \cos \beta)^2 + F &= 0
 \end{aligned}$$

Expanding this expression and collecting like terms (see Exercise 80), gives the following expressions for coefficients a , b , and c of the corresponding equation $aX^2 + bXY + cY^2 + f = 0$:

$$\begin{aligned}
 a &\rightarrow A \cos^2 \beta + B \sin \beta \cos \beta + C \sin^2 \beta && a \text{ is the coefficient of } X^2 \\
 b &\rightarrow -2A \sin \beta \cos \beta + B(\cos^2 \beta - \sin^2 \beta) + 2C \sin \beta \cos \beta && b \text{ is the coefficient of } XY \\
 c &\rightarrow A \sin^2 \beta - B \sin \beta \cos \beta + C \cos^2 \beta && c \text{ is the coefficient of } Y^2 \\
 f &\rightarrow F && f = F \text{ (the constant remains unchanged)}
 \end{aligned}$$

To accomplish our purpose, we require the coefficient b to be zero. While this expression looks daunting, the double-angle identities for sine and cosine simplify it very nicely:

$$\begin{aligned}
 b &\rightarrow -A(2 \sin \beta \cos \beta) + B(\cos^2 \beta - \sin^2 \beta) + C(2 \sin \beta \cos \beta) = 0 && (1) \\
 -A \sin(2\beta) + B \cos(2\beta) + C \sin(2\beta) &= 0 && (2) \\
 (C - A)\sin(2\beta) &= -B \cos(2\beta) && (3) \\
 \tan(2\beta) &= \frac{-B}{C - A} && (4) \\
 \tan(2\beta) &= \frac{B}{A - C}; A \neq C && (5)
 \end{aligned}$$

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Note from line (3) that $A = C$ would imply $\cos(2\beta) = 0$, giving $2\beta = 90^\circ$ or -90° , with $\beta = 45^\circ$ or -45° (for the sake of convenience, we select the angle in QI). This fact can many times be used to great advantage. If $A \neq C$, $\tan(2\beta) = \frac{B}{A - C}$ and we choose 2β between 0 and 180° so that β will be in the first quadrant [$0 < \beta < 90^\circ$].

The Equation of a Conic After Rotating the Axes

For a conic defined by $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ and its graph in the xy -plane, an angle β can be determined using $\tan(2\beta) = \frac{B}{A - C}$ and used in the rotation formulas to find a polynomial $aX^2 + cY^2 + dX + eY + f = 0$ in XY -plane, where the conic is either vertical or horizontal.

EXAMPLE 4 ▶ Rotating the Axes to Eliminate the Bxy -Term

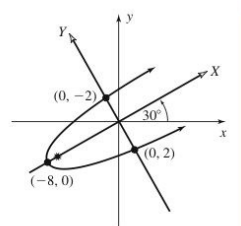
For $x^2 - 2\sqrt{3}xy + 3y^2 - \sqrt{3}x - y - 16 = 0$, eliminate the xy -term using a rotation of axes and identify the conic associated with the resulting equation. Then sketch the graph of the rotated conic in the XY -plane.

Solution ▶ Since $A \neq C$, we find β using $\tan(2\beta) = \frac{B}{A - C}$, giving $\tan(2\beta) = \frac{-2\sqrt{3}}{1 - 3} = \sqrt{3}$. This shows $2\beta = \tan^{-1}\sqrt{3}$, yielding $2\beta = 60^\circ$ so $\beta = 30^\circ$. Using $\cos 30^\circ = \frac{\sqrt{3}}{2}$ and $\sin 30^\circ = \frac{1}{2}$ along with the rotation formulas we obtain the following XY -equation, with corresponding terms shown side-by-side for clarity:

Given Term in xy -Plane	Corresponding Term in xy -Plane
x^2	$\rightarrow \left(\frac{\sqrt{3}}{2}X - \frac{1}{2}Y\right)^2 = \frac{3}{4}X^2 - \frac{\sqrt{3}}{2}XY + \frac{1}{4}Y^2$
$-2\sqrt{3}xy$	$\rightarrow -2\sqrt{3}\left(\frac{\sqrt{3}}{2}X - \frac{1}{2}Y\right)\left(\frac{1}{2}X + \frac{\sqrt{3}}{2}Y\right) = -\frac{3}{2}X^2 - \sqrt{3}XY + \frac{3}{2}Y^2$
$3y^2$	$\rightarrow 3\left(\frac{1}{2}X + \frac{\sqrt{3}}{2}Y\right)^2 = \frac{3}{4}X^2 + 3\frac{\sqrt{3}}{2}XY + \frac{9}{4}Y^2$
$-\sqrt{3}x$	$\rightarrow -\sqrt{3}\left(\frac{\sqrt{3}}{2}X - \frac{1}{2}Y\right) = -\frac{3}{2}X + \frac{\sqrt{3}}{2}Y$
$-y$	$\rightarrow -\left(\frac{1}{2}X + \frac{\sqrt{3}}{2}Y\right) = -\frac{1}{2}X - \frac{\sqrt{3}}{2}Y$
-16	$\rightarrow -16$

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Adding the like terms to the far right, the X^2 -terms (in red), the Y -terms (in bold), and the mixed XY -terms (in blue) sum to zero, leaving the equation $-2X + 4Y^2 - 16 = 0$, which is the parabola defined by $Y^2 = \frac{1}{2}(X + 8)$. This parabola is symmetric to the X -axis and opens to the right, with a vertex at $(-8, 0)$, Y -intercepts at $(0, -2)$ and $(0, 2)$, focus at $(-\frac{63}{8}, 0)$ and directrix through $(-\frac{65}{8}, 0)$. The graph is shown in the figure.



Now try Exercises 21 through 30 ►

In Example 4, the angle β was a **standard angle** and easily found. In general, this is not the case and finding exact values of $\cos \beta$ and $\sin \beta$ for use in the rotation formulas requires using $\tan(2\beta) = \frac{\sin(2\beta)}{\cos(2\beta)}$, the corresponding (triangle) diagram, and the identities

A. You've just learned how to graph conic sections that have nonvertical and nonhorizontal axes (rotated conics)

$\cos \beta = \sqrt{\frac{1 + \cos(2\beta)}{2}}$ and $\sin \beta = \sqrt{\frac{1 - \cos(2\beta)}{2}}$. See Exercises 31, 32, 84, and 85 for further study.

B. Identifying Conics Using the Discriminant

In addition to rotating the axes, the inclusion of the "xy-term" makes it impossible to identify the conic section using the tests seen earlier. For example, having $A = C$ no longer guarantees a circle, and $A = 0$ or $C = 0$ does not guarantee a parabola. Rather than continuing to look at what the mixed term and the resulting rotation *changes*, we now look at what the rotation *does not change*, called **invariants** of the transformation. These invariants can be used to double-check the algebra involved and to identify the conic using the **discriminant**. These are given here without proof.

Invariants of a Rotation and Classification Using the Discriminant

By rotating the coordinate axes through a predetermined angle β , the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ can be transformed into $aX^2 + cY^2 + dX + eY + f = 0$ in which the xy -term is absent. This rotation has the following invariants:

(1) $F = f$ (2) $A + C = a + c$ (3) $B^2 - 4AC = b^2 - 4ac$.

The discriminant of a conic equation in polynomial form is $B^2 - 4AC$. Except in degenerate cases, the graph of the equation can be classified as follows:

- If $B^2 - 4AC = 0$, the graph will be a parabola.
- If $B^2 - 4AC < 0$, the graph will be a circle or an ellipse.
- If $B^2 - 4AC > 0$, the graph will be a hyperbola.

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EXAMPLE 5A ▶ Verifying the Invariants of a Rotation of Axes

Verify the invariants just given using the equations from Example 4. Also verify the discriminant test.

Solution ▶ From the equation $x^2 - 2\sqrt{3}xy + 3y^2 - \sqrt{3}x - y - 16 = 0$, we have $A = 1$, $B = -2\sqrt{3}$, $C = 3$, $D = -\sqrt{3}$, $E = -1$, and $F = -16$. After applying the rotation the equation became $-2X + 4Y^2 - 16 = 0$, with $a = 0$, $b = 0$, $c = 4$, $d = -2$, $e = 0$, and $f = -16$. Checking each invariant gives (1) $-16 = -16\checkmark$, (2) $1 + 3 = 0 + 4\checkmark$, and (3) $(-2\sqrt{3})^2 - 4(1)(3) = (0)^2 - 4(0)(4)\checkmark$. With $B^2 - 4AC = 0$, the discriminant test indicates the conic is a parabola \checkmark .

EXAMPLE 5B ▶ Identifying the Equation of a Conic Using the Discriminant

Use the discriminant to identify each equation as that of a circle, ellipse, parabola, or hyperbola, but do not graph the equation.

- a. $3x^2 - 4xy + 3y^2 + 6x + 12y - 2 = 0$
- b. $4x^2 + 9xy + 4y^2 - 8x + 24y + 9 = 0$
- c. $6x^2 - 7xy + y^2 - 5 = 0$
- d. $x^2 - 6xy + 9y^2 + 6x = 0$

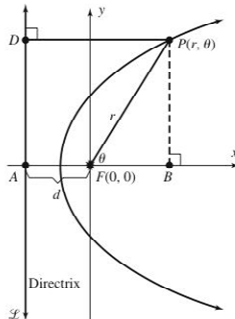
Solution ▶

<p>a. $A = 3; B = -4; C = 3$ $B^2 - 4AC = (-4)^2 - 4(3)(3)$ $= -20$ circle or ellipse</p>	<p>b. $A = 4; B = 9; C = 4$ $B^2 - 4AC = (9)^2 - 4(4)(4)$ $= 17$ hyperbola</p>
<p>c. $A = 6; B = -7; C = 1$ $B^2 - 4AC = (-7)^2 - 4(6)(1)$ $= 25$ hyperbola</p>	<p>d. $A = 1; B = -6; C = 9$ $B^2 - 4AC = (-6)^2 - 4(1)(9)$ $= 0$ parabola</p>

B. You've just learned how to identify conics using the discriminant of the polynomial form—the invariant $B^2 - 4AC$

Now try Exercises 33 through 36 ▶

Figure 10.52



C. Conic Equations in Polar Form

You might recall that earlier in this chapter we defined ellipses and hyperbolas in terms of a distance between two points, but a parabola in terms of a distance between a point and a line (the focus and directrix). Actually, all conic sections can be defined using a focus/directrix development and written in polar form. This serves to unify and greatly simplify their study. We begin by revisiting the focus/directrix development of a parabola, using a directrix \mathcal{L} and placing the focus at the origin. With the polar axis as the axis of symmetry and the point $P(r, \theta)$ in polar coordinates, we obtain the graph shown in Figure 10.52. Given D and A are points on \mathcal{L} (with A on the polar axis), we note the following:

- (1) $\overline{DP} = \overline{FP}$ definition of a parabola
- (2) $\overline{DP} = \overline{AB}$ equal line segments
- (3) $\overline{FB} = r \cos \theta$ $\cos \theta = \frac{\overline{FB}}{r}$
- (4) $\overline{AB} = \overline{AF} + \overline{FB}$ sum of line segments

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Using the preceding equations and representing the distance \overline{AF} by the constant d , we obtain this sequence:

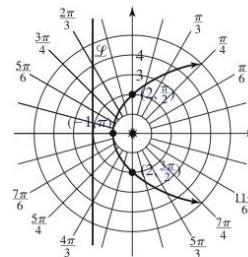
$$\begin{aligned} \overline{AB} &= d + r \cos \theta && \text{substitute } d \text{ for } \overline{AF} \text{ and } r \cos \theta \text{ for } \overline{FB} \\ \overline{FP} &= d + r \cos \theta && \text{substitute } \overline{FP} \text{ for } \overline{AB} \text{ since } \overline{FP} = \overline{DP} = \overline{AB} \\ r &= d + r \cos \theta && \text{substitute } r \text{ for } \overline{FP} \end{aligned}$$

Solving the last equation for r we have $r - r \cos \theta = d$, then $r = \frac{d}{1 - \cos \theta}$, which is the equation of a parabola in polar form with its focus at the origin, vertex at $(-\frac{d}{2}, \pi)$, and y -intercepts at $(d, \frac{\pi}{2})$ and $(d, \frac{3\pi}{2})$. Note the constant "1" in the denominator is a key characteristic of polar equations, and helps define the standard form.

EXAMPLE 6A ▶ Identifying a Conic from Its Polar Equation

Verify the equation $r = \frac{6}{3 - 3 \cos \theta}$ represents a parabola, then describe and sketch the graph.

Solution ▶ Write the equation in standard form by dividing the numerator and denominator by 3, obtaining $r = \frac{2}{1 - \cos \theta}$. From this we see $d = 2$ and the represents a parabola symmetric to the polar axis, with vertex at $(-1, \pi)$ and y -intercepts at $(2, \frac{\pi}{2})$ and $(2, \frac{3\pi}{2})$, as shown in the figure.



The polar equation for a parabola depended on $\frac{FP}{DP}$ and \overline{FP} being equal in length, with ratio $\frac{FP}{DP} = 1$. But what if this ratio is not equal to 1?

Similar to our introduction to conics in Section 10.1, we assume $\frac{FP}{DP} = \frac{1}{2}$ and investigate the graph that

results. Cross-multiplying gives $2\overline{FP} = \overline{DP}$, which states that the distance from D to P is twice the distance from F to P . Note that we are able to locate two points P_1 and P_2 on the polar axis that satisfy this relation, rather than only one as in the case of the parabola. Figure 10.53 illustrates the location of these points. Using the focal chord for convenience, two additional points P_3 and P_4 can be located that also satisfy the stated condition (see Figure 10.54). In fact, we can locate an infinite number of these points using $\frac{FP}{DP} = \frac{1}{2}$, and the resulting graph appears to be an ellipse (and is definitely not a parabola). These illustrations provide the basis for stating a general focus/directrix definition of the conic sections. The ratio $\frac{FP}{DP}$ is often represented by the letter e , and represents the **eccentricity** of the conic. Using $\overline{FP} = r$ and $\overline{DP} = d + r \cos \theta$ from our initial development, $\frac{FP}{DP} = \frac{r}{d + r \cos \theta} = e$, which enables us to state the general

Figure 10.53

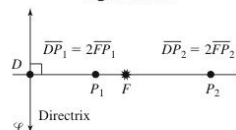
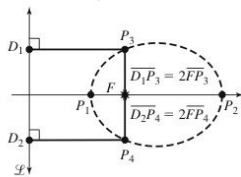


Figure 10.54



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equation of a conic in polar form. Solving for r leads to the equation $r = \frac{de}{1 - e \cos \theta}$, where the type of conic depends solely on e . Depending on the orientation of the conic, the general form may involve sine instead of cosine, and have a sum of terms in the denominator rather than a difference. Note once again that if $e = 1$, the relation simplifies into the parabolic equation seen earlier.

The Standard Equation of a Conic in Polar Form

Given a conic section with eccentricity e , one foci at the pole of the $r\theta$ -plane, and directrix \mathcal{L} located d units from this focus. Then the polar equations

$$r = \frac{de}{1 \pm e \cos \theta} \quad \text{and} \quad r = \frac{de}{1 \pm e \sin \theta}$$

represent one of the conic sections as determined by the value of e .

- If $e = 1$, the graph is a parabola.
- If $0 < e < 1$, the graph is an ellipse.
- If $e > 1$, the graph is a hyperbola.

For the ellipse and hyperbola, the major axis and transverse axis (respectively) are both perpendicular to the directrix and contain the vertices and foci. Our earlier development of eccentricity can then be expressed in terms of a and c , as the ratio $e = \frac{c}{a}$.

As in our previous study of polar equations, if the equation involves cosine the graph will be symmetric to the polar axis. If the graph involves sine, the line $\theta = \frac{\pi}{2}$ is the axis of symmetry. In addition, if the denominator contains a difference of terms (as in Example 6A), the graph will be above or to the right of the directrix (depending on whether the equation involves sine or cosine). If the denominator contains a sum of terms, the graph will be below or to the left of the directrix.

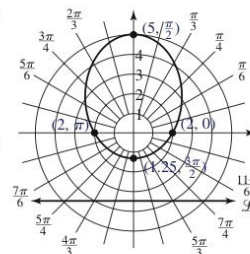
EXAMPLE 6B ▶ Using the Standard Equation to Graph a Conic in Polar Form

Determine if the equation $r = \frac{10}{5 - 3 \sin \theta}$ represents a parabola, ellipse, or hyperbola. Then describe and sketch the graph.

Solution ▶ To write the equation in standard form, we divide both numerator and denominator by 5, obtaining the equation $r = \frac{2}{1 - \frac{3}{5} \sin \theta}$. From the standard

form we note $e = \frac{3}{5}$ so the equation represents an ellipse. With a difference of terms and the sine function involved, the graph is symmetric to $\theta = \frac{\pi}{2}$ and is above the directrix. Given so much information by the equation, we require very few points to sketch the graph and settle for those generated by

$\theta = 0, \frac{\pi}{2}, \pi,$ and $\frac{3\pi}{2}$, yielding the points $(2, 0), (5, \frac{\pi}{2}), (2, \pi)$, and $(\frac{5}{4}, \frac{3\pi}{2})$. The graph is shown in the figure.



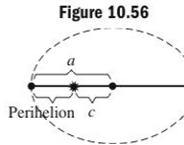
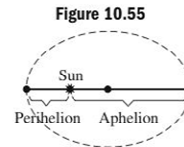
✓ **C.** You've just learned how to write the equation of a conic section in polar form

Now try Exercises 37 through 56 ▶

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D. Applications of Conics in Polar Form

For centuries it has been known that the orbits of the planets around the Sun are elliptical, with the Sun at one focus. In addition, comets may approach our Sun in an elliptical, hyperbolic, or parabolic path with the Sun again at the foci. This makes planetary studies a very natural application of the conic sections in polar form. To aid this study, it helps to know that in an elliptical orbit, the maximum distance of a planet from the Sun is called its **aphelion**, and the shortest distance is the **perihelion** (Figure 10.55). This means the length of the major axis is “aphelion + perihelion,” enabling us to find the value of c if the aphelion and perihelion are known (Figure 10.56). Using $e = \frac{c}{a}$, we can then find the eccentricity of the planet’s orbit.



EXAMPLE 7 ▶ Determining the Eccentricity of a Planet’s Orbit

In its elliptical orbit around the Sun, Mars has an aphelion of 154.9 million miles and a perihelion of 128.4 million miles. What is the eccentricity of its orbit?

Solution ▶ The length of the major axis would be $2a = (154.9 + 128.4)$ mi, yielding a semimajor axis of $a = 141.65$ million miles. Since $a = c + \text{perihelion}$ (Figure 10.56), we have $141.65 = c + 128.4$ so $c = 13.25$. The eccentricity of the orbit is $e = \frac{c}{a} = \frac{13.25}{141.65}$ or about 0.0935.

Now try Exercises 59 and 60 ▶

We can also find the perihelion and aphelion directly in terms of a (semimajor axis) and e (eccentricity) if these quantities are known. Using $a = c + \text{perihelion}$, we obtain: $\text{perihelion} = a - c$. For $e = \frac{c}{a}$, we have $ea = c$ and by direct substitution we obtain: $\text{perihelion} = a - ea = a(1 - e)$. For Example 8, recall that “AU” designates an *astronomical unit*, and represents the mean distance from the Earth to the Sun, approximately 92.96 million miles.

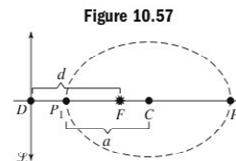
EXAMPLE 8 ▶ Determining the Perihelion of a Planet’s Orbit

The orbit of the planet Jupiter has a semimajor axis of 5.2 AU (1 AU \approx 92.96 million miles) and an eccentricity of 0.0489. What is the closest distance from Jupiter to the Sun?

Solution ▶ With $\text{perihelion} = a(1 - e)$, we have $5.2(1 - 0.0489) \approx 4.946$. At its closest approach, Jupiter is 4.946 AU from the Sun (about 460 million miles).

Now try Exercises 61 through 64 ▶

To find the polar equation of a planetary orbit, it’s helpful to write the general polar equation in terms of the semimajor axis a , which is often known or easily found, rather than in terms of the distance d from directrix to focus, which is often unknown. Consider the diagram in Figure 10.57, which shows an elliptical orbit with the Sun at one focus, vertices



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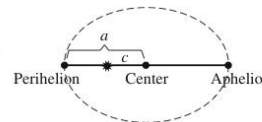
P_1 and P_2 (perihelion and aphelion), and the center C of the ellipse. Assume the point P used to define the conic sections is at position P_1 , giving $\frac{FP_1}{DP_1} = e$. From Example 8 we have $\overline{FP_1} = a(1 - e)$. Substituting $a(1 - e)$ for $\overline{FP_1}$ and solving for $\overline{DP_1}$ gives $\overline{DP_1} = \frac{a(1 - e)}{e}$. Using $d = \overline{DP_1} + \overline{FP_1}$, we obtain the following sequence:

$$\begin{aligned} d &= \overline{DP_1} + \overline{FP_1} \\ &= \frac{a(1 - e)}{e} + a(1 - e) && \text{substitute } \frac{a(1 - e)}{e} \text{ for } \overline{DP_1} \text{ and } a(1 - e) \text{ for } \overline{FP_1} \\ &= \frac{a(1 - e)}{e} + \frac{ae(1 - e)}{e} && \text{common denominator} \\ &= \frac{a(1 - e)(1 + e)}{e} && \text{combine terms, factor out } a(1 - e) \\ &= \frac{a(1 - e^2)}{e} && (1 - e)(1 + e) = 1 - e^2 \\ de &= a(1 - e^2) && \text{multiply by } e \end{aligned}$$

Substituting $a(1 - e^2)$ for de in the standard equation $r = \frac{de}{1 - e \cos \theta}$ gives the equation of the orbit entirely in terms of a and e : $r = \frac{a(1 - e^2)}{1 - e \cos \theta}$

EXAMPLE 9 ▶ Writing the Polar Equation of an Ellipse from Given Information

At its aphelion, the dwarf planet Pluto is the most distant from the Sun at 4538 million miles. It has a perihelion of 2756 million miles. Use this information to find the polar equation that models the orbit of Pluto, then find the length of the focal chord for this ellipse.



Solution ▶ With all figures in millions of miles, the major axis is $2a = 4538 + 2756 = 7294$, so the semimajor axis has length $a = 3647$. With $a = c + \text{perihelion}$, we obtain $3647 = c + 2756$ or $c = 891$. The eccentricity of the orbit is $e = \frac{891}{3647} \approx 0.2443$.

The polar equation for the orbit of Pluto is $r \approx \frac{(3647)(1 - [0.2443]^2)}{1 - [0.2443] \cos \theta}$ or $r \approx \frac{3430}{1 - 0.2443 \cos \theta}$. Substituting $\theta = \frac{\pi}{2}$ (since the left-most focus is at the pole), we obtain $r = 3430$, so the length of the focal chord is $2(3430) = 6860$ million miles.

D. you've just learned how to solve applications involving the conic sections in polar form

Now try Exercises 65 through 70 ▶

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TECHNOLOGY HIGHLIGHT

Investigating the Eccentricity e

One meaning of the word eccentric is “to deviate from a circular pattern.” In a very real sense, this is the role that eccentricity plays as it helps to describe the conic sections. For an ellipse we’ve learned that $0 < e < 1$. If the eccentricity is near zero, there is little deviation and the ellipse appears nearly circular. If e is near 1, the ellipse is very elongated. To explore the eccentricity of an ellipse, enter the equation $r = \frac{a(1 - e^2)}{1 - e \cos \theta}$ on the **Y=** screen, using $a = 2$ (arbitrarily chosen) and **ALPHA SIN** “E” for the eccentricity. The result is shown in Figure 10.58. We will enter and store values for E on the home screen and graph the resulting ellipse (see Exercise 2 for an alternative method). Return to the home screen and enter 0.1 **STO >** **ALPHA SIN** and graph the result on the **ZOOM 4:ZDecimal** screen. Repeat the procedure using $e = 0.25, 0.5, 0.75,$ and 0.9 . The graphs for $e = 0.1$ and $e = 0.9$ are shown in Figures 10.59 and 10.60. As you can see, when $e = 0.1$ the ellipse is nearly circular, while $e = 0.9$ produces a graph that is cigar shaped.

Figure 10.58

```

Plot1 Plot2 Plot3
r1=2(1-E^2)/(1-E
COS(theta))
r2=
r3=
r4=
r5=
r6=
                    
```

Figure 10.59

Figure 10.60

Exercise 1: Try entering a value of $e = 0$, then use your graphing calculator and basic knowledge to verify the resulting graph is a circle.

Exercise 2: Try the same exercise using the set/list option. In other words, enter the equation as shown here, with the values of e in braces $\{ \}$: $r_1 = \frac{2(1 - \{0.1, 0.25, 0.5, 0.75, 0.9\}^2)}{(1 - \{0.1, 0.25, 0.5, 0.75, 0.9\}\cos(\theta))}$. This will enable you to view all five ellipses on the same screen. Discuss the similarities and differences of this family of graphs.

10.6 EXERCISES

► **CONCEPTS AND VOCABULARY**

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The set of points (x, y) in the xy -plane are related to points (X, Y) in the XY -plane by the _____ formulas. To find the angle β between the original axes and the rotated axes, we use $\tan(2\beta) = \frac{2B}{A-C}$.

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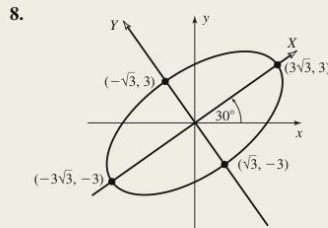
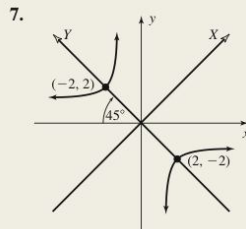
2. For a point P on the graph of a conic with focus F and D a point on the directrix, the ratio $\frac{FP}{DP}$ gives the _____ of the graph. For the eccentricity e , if $e = 1$ the graph is a _____, if $e > 1$ the graph is a _____, and if $0 < e < 1$ the graph will be an ellipse.
3. Features or relationships that do not change when certain transformations are applied are called _____ of the transformation.
4. The _____ form of the equation of a conic is $r = \frac{de}{1 \pm e \cos \theta}$ if the graph is symmetric to

the _____ axis, and $r = \frac{de}{1 \pm e \sin \theta}$ if symmetric to the line _____.

5. Discuss the advantages of graphing a rotated conic using the rotation of axes, over graphing by simply plotting points.
6. Discuss the primary advantages of using $r = \frac{a(1 - e^2)}{1 - e \cos \theta}$ rather than $r = \frac{de}{1 - e \cos \theta}$ to develop the equation of planetary orbit.

► DEVELOPING YOUR SKILLS

The graph of a conic rotated in the xy -plane is given. Use the graph (not the rotation of axes formulas) to find the equation of the conic in the XY -plane.



Given the point (x, y) in the xy -plane, find the coordinates of this point in the XY -plane given the angle β between the xy -axes and the XY -axes is 45° .

9. $(6\sqrt{2}, 6)$
10. $(4, 3\sqrt{2})$
11. $(0, 5)$
12. $(8, 0)$

Given the point (X, Y) in the XY -plane, find the coordinates of this point in the xy -plane given the angle β between the xy -axes and the XY -axes is 30° .

13. $(2, 2\sqrt{3})$
14. $(\sqrt{3}, 3)$
15. $(3, 4)$
16. $(12, 5)$

The conic sections whose equations are given in the XY -plane are rotated clockwise by the indicated angle. Find the corresponding equation in the xy -plane.

17. $X^2 - Y^2 = 9$; 60°
18. $X^2 + Y = 4$; 60°

The conic sections whose equations are given in the xy -plane are rotated by the indicated angle. What is the corresponding equation in the XY -plane?

19. $3x^2 + 2xy + 3y^2 = 9$; 45°
20. $x^2 + \sqrt{3}xy + 2y^2 = 8$; 60°

For the given conics in the xy -plane, (a) use a rotation of axes to find the corresponding equation in the XY -plane (clearly state the angle of rotation β), and (b) sketch its graph. Be sure to indicate the characteristic features of each conic in the XY -plane.

21. $x^2 + 4xy + y^2 - 2 = 0$
22. $x^2 + 2xy + y^2 - 12 = 0$
23. $5x^2 + 6xy + 5y^2 = 16$
24. $5x^2 - 26xy + 5y^2 = -72$
25. $x^2 + 10\sqrt{3}xy + 11y^2 = -64$
26. $37x^2 + 42\sqrt{3}xy + 79y^2 - 400 = 0$

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10-73

Section 10.6 More on the Conic Sections: Rotation of Axes and Polar Form

991

27. $3x^2 - 2\sqrt{3}xy + y^2 - 8x - 8\sqrt{3}y = 0$

28. $6x^2 - 4\sqrt{3}xy + 2y^2 + 2x + 2\sqrt{3}y = 0$

29. $13x^2 - 6\sqrt{3}xy + 7y^2 - 100 = 0$

30. $x^2 + 4xy + y^2 + \sqrt{2}x + \sqrt{2}y = -11$

Identify the graph of each equation using the discriminant, then find the value of $\cos(2\beta)$ using $\tan(2\beta) = \frac{\sin(2\beta)}{\cos(2\beta)}$ and the related triangle diagram.

Finally, find $\sin \beta$ and $\cos \beta$ using the half-angle identities $\cos \beta = \sqrt{\frac{1 + \cos(2\beta)}{2}}$ and

$\sin \beta = \sqrt{\frac{1 - \cos(2\beta)}{2}}$.

31. $12x^2 + 24xy + 5y^2 - 40x - 30y = 25$

32. $25x^2 + 840xy - 16y^2 - 400 = 0$

For the following equations, (a) use the discriminant to identify the equation as that of a circle, ellipse, parabola, or hyperbola; (b) find the angle of rotation β and use it to find the corresponding equation in the XY -plane; and (c) verify all invariants of the transformation.

33. $x^2 - 2xy + y^2 - 5 = 0$

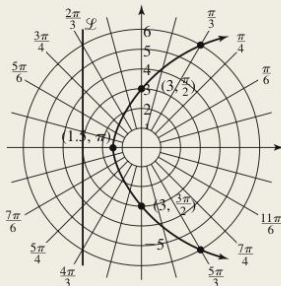
34. $2x^2 - 3xy + 2y^2 = 0$

35. $3x^2 + \sqrt{3}xy + 4y^2 + 4x = 1$

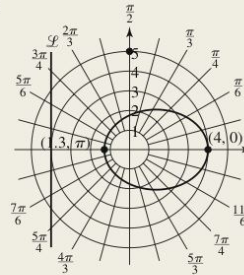
36. $3x^2 + 8\sqrt{3}xy - 5y^2 + 12y = -2$

Match each graph to its corresponding equation. Justify your answers (two equations have no match).

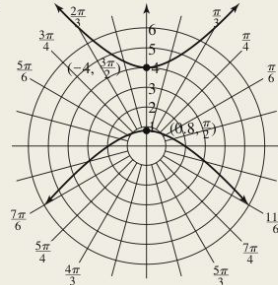
37.



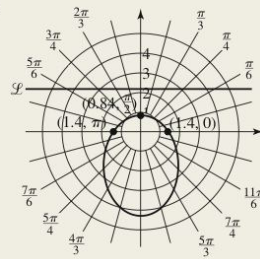
38.



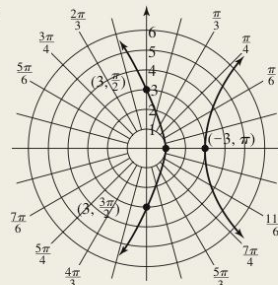
39.



40.

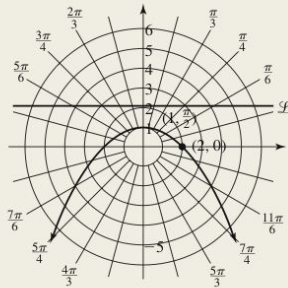


41.



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42.



- a. $r = \frac{10}{5 + 5 \sin \theta}$
- b. $r = \frac{8}{4 - 2 \cos \theta}$
- c. $r = \frac{5.4}{3 - 2 \sin \theta}$
- d. $r = \frac{4.2}{3 + 2 \sin \theta}$
- e. $r = \frac{12}{4 + 6 \sin \theta}$
- f. $r = \frac{6}{2 - 2 \cos \theta}$
- g. $r = \frac{4}{2 + 3 \sin \theta}$
- h. $r = \frac{9}{3 + 6 \cos \theta}$

For the conic equations given, determine if the equation represents a parabola, ellipse, or hyperbola. Then describe and sketch the graphs using polar graph paper.

- 43. $r = \frac{4}{2 + 2 \sin \theta}$
- 44. $r = \frac{10}{5 - 5 \sin \theta}$
- 45. $r = \frac{12}{6 - 3 \sin \theta}$
- 46. $r = \frac{6}{4 + 3 \cos \theta}$
- 47. $r = \frac{6}{2 + 4 \cos \theta}$
- 48. $r = \frac{2}{2 - 3 \sin \theta}$
- 49. $r = \frac{5}{5 + 4 \cos \theta}$
- 50. $r = \frac{2}{4 - 5 \sin \theta}$

Write the equation of a conic that satisfies the conditions given. Assume each has one focus at the pole.

- 51. ellipse, $e = 0.8$, directrix to focus: $d = 4$
- 52. hyperbola, $e = 1.25$, directrix to focus: $d = 6$
- 53. parabola, vertex at $(2, \pi)$
- 54. ellipse, $e = 0.35$, vertex at $(4, 0)$
- 55. hyperbola, $e = 1.5$, vertex at $(3, \frac{\pi}{2})$
- 56. parabola, directrix to focus: $d = 5.4$

► WORKING WITH FORMULAS

57. Equation of a line in polar form:

$$r = \frac{C}{A \cos \theta + B \sin \theta}$$

For the line $Ax + By = C$ in the xy -plane with slope $m = -\frac{A}{B}$ and y -intercept $(0, \frac{C}{B})$, the

corresponding equation in the $r\theta$ -plane is given by the formula shown. (a) Given the line $2x + 3y = 12$ in the xy -plane, find the corresponding polar equation and (b) verify

that $-\frac{A}{B} = -\frac{r(\pi/2)}{r(0)}$.

58. Polar form of an ellipse with center at the pole:

$$r^2 = \frac{a^2 b^2}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$$

If an ellipse in the $r\theta$ -plane has its center at the pole (with major axis parallel to the x -axis), its equation is given by the formula here, where $2a$ and $2b$ are the lengths of the major and minor axes, respectively. (a) Given an ellipse with center at the pole has a major axis of length 8 and a minor axis of length 4, find the equation of the ellipse in polar form and (b) graph the result on a calculator and verify that $2a = 8$ and $2b = 4$.

► APPLICATIONS

Planetary motion: The perihelion, aphelion, and orbital period of the planets Jupiter, Saturn, Uranus, and Neptune are shown in the table. Use the information to answer or complete the following exercises. The formula $L = 2\pi\sqrt{0.5(a^2 + b^2)}$ can be used to estimate the length of the orbital path. Recall for an ellipse, $c^2 = a^2 - b^2$.

Planet	Perihelion (10 ⁶ mi)	Aphelion (10 ⁶ mi)	Period (yr)
Jupiter	460	507	11.9
Saturn	840	941	29.5
Uranus	1703	1866	84
Neptune	2762	2824	164.8

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10-75

Section 10.6 More on the Conic Sections: Rotation of Axes and Polar Form

993

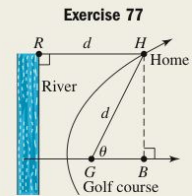
- 59. Find the eccentricity of the planets Jupiter and Saturn.
- 60. Find the eccentricity of the planets Uranus and Neptune.
- 61. The orbit of Pluto (a dwarf planet) has a semimajor axis of 3647 million miles and an eccentricity of $e = 0.2443$. Find the perihelion of Pluto.
- 62. The orbit of Ceres (a large asteroid) has a semimajor axis 257 million miles and an eccentricity of $e = 0.097$. Find the perihelion of Ceres.
- 63. Which of the four planets in the table given has the greatest orbital eccentricity?
- 64. Which of these four planets has the greatest orbital velocity?
- 65. Find the polar equation modeling the orbit of Jupiter.
- 66. Find the polar equation modeling the orbit of Saturn.
- 67. Find the polar equation modeling the orbit of Uranus.
- 68. Find the polar equation modeling the orbit of Neptune.
- 69. Suppose all four major planets arrived at the focal chord of their orbit ($\theta = \frac{\pi}{2}$) simultaneously. Use the equations in Exercises 65 to 68 to determine the distance between each of the planets at this moment.
- 70. The polar equation for the orbit of Pluto (a dwarf planet) was developed in Example 9. From an earlier exercise, the polar equation for the orbit of Neptune is $r \approx \frac{2793}{1 - 0.0111 \cos \theta}$. Using the TABLE of your graphing calculator, determine if Pluto is *always* the farthest planet from the Sun. If not, how much further from the Sun is Neptune than Pluto at their perihelion?

Mirror manufacturing: A modern manufacturer of oval (elliptical) mirrors for consumer use has programmed the equipment to automatically cut the glass for each mirror (major axis horizontal). The most popular mirrors are those that fit within a golden rectangle (ratio of L to W is approximately 1 to 0.618). Find the polar equation the manufacturer should use to program the equipment for mirror orders of the following lengths. Recall that $c^2 = a^2 - b^2$ and $e = \frac{c}{a}$ and assume one focus is at the pole.

- 71. $L = 4$ ft
- 72. $L = 3.5$ ft

- 73. $L = 1.5$ m
- 74. $L = 0.5$ m
- 75. Referring to Exercises 71 to 74, find the total cost of each mirror (to the consumer) if they sell for \$75 per square foot (\$807 per square meter). The area of an ellipse is given by $A = \pi ab$.
- 76. Referring to Exercises 71 to 74, find the total cost of an elliptical frame for each mirror (to the consumer) if the frame sells for \$12.50 per linear foot (\$41.01 per meter). The circumference of an ellipse is approximated by $C = \pi \sqrt{2(a^2 + b^2)}$.

77. Home location: Candice is an enthusiastic golfer and an avid swimmer. After being transferred to a new city, she decides to buy a house that is an equal distance from the local golf course and the river running through the city. If the distance



- between the river and the golf course at the closest point is 3 mi, find the polar equation of the parabola that will trace through the possible locations for her new home. Assume the golf course is at the focus of the parabola.
- 78. **Home location:** Referring to Exercise 77, assume Candice finds the perfect dream house in a subdivision located at $(6, \frac{\pi}{3})$. Does this home fit the criteria (is it an equal distance from the river and golf course)?
- 79. Solve the system below for y to verify the rotation formula for y given on page 980.

$$\begin{cases} X = x \cos \beta + y \sin \beta \\ Y = y \cos \beta - x \sin \beta \end{cases}$$

80. Rotation of a conic section: Expand the following, collect like terms, and simplify. Show the result is the equation $aX^2 + bXY + cY^2 + f = 0$, where the coefficients $a, b, c,$ and f are as given on page 981. $A(X \cos \beta - Y \sin \beta)^2 + B(X \cos \beta - Y \sin \beta)(X \sin \beta + Y \cos \beta) + C(X \sin \beta + Y \cos \beta)^2 + F = 0$

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▶ EXTENDING THE CONCEPT

81. Using the rotation of axes formulas in the general equation $Ax^2 + Bxy + Cy^2 + F = 0$ ($D = E = 0$), we were able to obtain the equation $aX^2 + bXY + cY^2 + f = 0$ (see page 981), where

$$\begin{aligned} a &\rightarrow A \cos^2\beta + B \sin\beta \cos\beta + C \sin^2\beta \\ b &\rightarrow -2A \sin\beta \cos\beta + B(\cos^2\beta - \sin^2\beta) + 2C \sin\beta \cos\beta \\ c &\rightarrow A \sin^2\beta - B \sin\beta \cos\beta + C \cos^2\beta \text{ and } f \rightarrow F \end{aligned}$$

- a. Use these to verify $b^2 - 4ac = B^2 - 4AC$. b. Use these to verify $a + c = A + C$. c. Explain why the invariant $f = F$ must always hold.
82. A short-period comet is one that orbits the Sun in 200 yr or less. Two of the best known are Halley's Comet and Encke's Comet. Using any of the resources available to you, find the perihelion and aphelion of each comet and use the information to find the lengths of the semimajor and semiminor axes. Also find the period of each comet. If the length of an elliptical (orbital) path is approximated by $L = 2\pi\sqrt{0.5(a^2 + b^2)}$, find the approximate average speed of each comet in miles per hour. Finally, determine the polar equation of each orbit.

For the given conics in the xy -plane, use a rotation of axes to find the corresponding equation in the XY -plane. See Exercises 31 and 32.

83. In the $r\theta$ -plane, the equation of a circle having radius R , center at (R, β) , and going through the pole is given by $r = 2R \cos(\theta - \beta)$. Consider the circle defined by $x^2 + y^2 - 6\sqrt{2}x - 6\sqrt{2}y = 0$ in the xy -plane. Verify this circle goes through the origin, then find the equation of the circle in polar form.
84. $12x^2 + 24xy + 5y^2 - 40x - 30y = 25$
85. $25x^2 + 840xy - 16y^2 - 400 = 0$
86. A right triangle in the xy -plane had vertices at $(0, 0)$, $(8, 0)$, and $(8, 6)$. Use the matrix equation $\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ to find the vertices in the XY -plane after the triangle is rotated 60° .
87. A square in the XY -plane has vertices at $(0, 0)$, $(2\sqrt{3}, 2)$, $(2\sqrt{3} - 2, 2 + 2\sqrt{3})$ and $(-2, 2\sqrt{3})$. Use the matrix equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix}$ to find the vertices in the xy -plane after the triangle is rotated -30° .

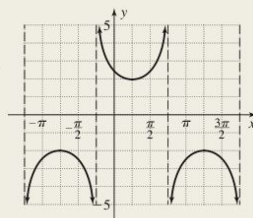
▶ MAINTAINING YOUR SKILLS

88. (8.2) Solve the system using elimination.

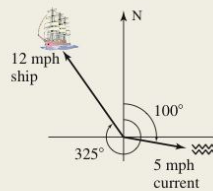
$$\begin{cases} x + 2y - z = -3 \\ -2x - 6y + z = 4 \\ 5x + 4y - 2z = -3 \end{cases}$$

89. (4.5) Solve for x (to the nearest tenth): $21.7 = 77.5e^{-0.0052x} - 44.95$

90. (5.5) Use the graph shown to write an equation of the form $y = A \sec(Bx + C)$. Clearly state the values of A , B , and C .



91. (7.3) A ship is moving at 12 mph on a heading of 325° , with a 5 mph current flowing at a 100° heading. Find the true course and speed of the ship.



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10.7 Parametric Equations and Graphs

Learning Objectives

In Section 10.7 you will learn how to:

- A. Sketch the graph of a parametric equation
- B. Write parametric equations in rectangular form
- C. Graph curves from the cycloid family
- D. Solve applications involving parametric equations

A large portion of the mathematics curriculum is devoted to functions, due to their overall importance and widespread applicability. But there are a host of applications for which nonfunctions are a more natural fit. In this section, we show that many *non-functions* can be expressed as **parametric equations**, where each is actually a *function*. These equations can be appreciated for the diversity and versatility they bring to the mathematical spectrum.

A. Sketching a Curve Defined Parametrically

Suppose you were given the set of points in the table here, and asked to come up with an equation model for the data. To begin, you might plot the points to see if any patterns or clues emerge, but in this case the result seems to be a curve we've never seen before (see Figure 10.61).

x	0	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0
y	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Figure 10.61

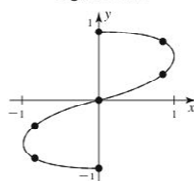
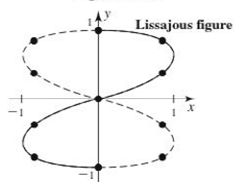


Figure 10.62



You also might consider running a regression on the data, but it's not possible since the graph is obviously not a function. However, a closer look at the data reveals the y -values could be modeled *independently of the x -values* by a cosine function, $y = \cos t$ for $t \in [0, \pi]$. This observation leads to a closer look at the x -values, which we find could be modeled by a sine function over the same interval, namely, $x = \sin(2t)$ for $t \in [0, \pi]$. These two functions combine to name all points on this curve, and both use the independent variable t called a **parameter**. The functions $x = \sin(2t)$ and $y = \cos t$ are called the parametric equations for this curve. The complete curve, shown in Figure 10.62, is called a **Lissajous figure**, or a closed graph (coincident beginning and ending points) that crosses itself to form two or more loops. Note that since the maximum value of x and y is 1 (the amplitude of each function), the entire figure will fit within a 1×1 rectangle centered at the origin. This observation can often be used to help sketch parametric graphs with trigonometric parameters. In general, parametric equations can take many forms, including polynomial, exponential, trigonometric, and other forms.

Parametric Equations

Given the set of points $P(x, y)$ such that $x = f(t)$ and $y = g(t)$, where f and g are both defined on an interval of the domain, the equations $x = f(t)$ and $y = g(t)$ are called parametric equations, with parameter t .

EXAMPLE 1 ▶ Graphing a Parametric Curve Where f and g Are Algebraic

Graph the curve defined by the parametric equations $x = t^2 - 3$ and $y = 2t + 1$.

Solution ▶ Begin by creating a table of values using $t \in [-3, 3]$. After plotting ordered pairs (x, y) , the result appears to be a parabola, opening to the right.

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1094

Coburn: Algebra and Trigonometry, Second Edition

10. Analytic Geometry and the Conic Sections

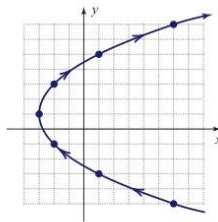
10.7: Parametric Equations and Graphs

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996

CHAPTER 10 Analytic Geometry and the Conic Sections

10-78



t	$x = t^2 - 3$	$y = 2t + 1$
-3	6	-5
-2	1	-3
-1	-2	-1
0	-3	1
1	-2	3
2	1	5
3	6	7

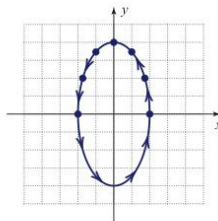
Now try Exercises 7 through 12, Part a ▶

If the parameter is a trig function, we'll often use standard angles as inputs to simplify calculations and the period of the function(s) to help sketch the resulting graph. Also note that successive values of t give rise to a directional evolution of the graph, meaning the curve is traced out in a direction dictated by the points that correspond to the next value of t . The arrows drawn along the graph illustrate this direction, also known as the **orientation** of the graph.

EXAMPLE 2 ▶ Graphing a Parametric Curve Where t and g Are Trig Functions

Graph the curve defined by the parametric equations $x = 2 \cos t$ and $y = 4 \sin t$.

Solution ▶ Using standard angle inputs and knowing the maximum value of any x - and y -coordinate will be 2 and 4, respectively, we begin computing and graphing a few points. After going from 0 to π , we note the graph appears to be a vertical ellipse. This is verified using standard values from π to 2π . Plotting the points and connecting them with a smooth curve produces the ellipse shown in the figure.



t	$x = 2 \cos t$	$y = 4 \sin t$
0	2	0
$\frac{\pi}{6}$	$\sqrt{3}$	2
$\frac{\pi}{3}$	1	$2\sqrt{3}$
$\frac{\pi}{2}$	0	4
$\frac{2\pi}{3}$	-1	$2\sqrt{3}$
$\frac{5\pi}{6}$	$-\sqrt{3}$	2
π	-2	0

Now try Exercises 13 through 18, Part a ▶

✓ **A.** You've just learned how to sketch the graph of a parametric equation

Note the ellipse has a counterclockwise orientation.

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B. Writing Parametric Equations in Rectangular Form

When graphing parametric equations, there are sometimes alternatives to simply plotting points. One alternative is to try and *eliminate the parameter*, writing the parametric equations in standard, rectangular form. To accomplish this we use some connection that allows us to “rejoin” the parameterized equations, such as variable t itself, a trigonometric identity, or some other connection.

EXAMPLE 3 ▶ Eliminating the Parameter to Obtain the Rectangular Form

Eliminate the parameter from the equations in Example 1: $x = t^2 - 3$ and $y = 2t + 1$.

Solution ▶ Solving for t in the second equation gives $t = \frac{y-1}{2}$, which we then substitute into the first. The result is $x = \left(\frac{y-1}{2}\right)^2 - 3 = \frac{1}{4}(y-1)^2 - 3$. Notice this is indeed a horizontal parabola, opening to the right, with vertex at $(-3, 1)$.

Now try Exercises 7 through 12, Part b ▶

EXAMPLE 4 ▶ Eliminating the Parameter to Obtain the Rectangular Form

Eliminate the parameter from the equations in Example 2: $x = 2 \cos t$ and $y = 4 \sin t$.

Solution ▶ Instead of trying to solve for t , we note the parametrized equations involve sine and cosine functions with the same argument (t), and opt to use the identity $\cos^2 t + \sin^2 t = 1$. Squaring both equations and solving for $\cos^2 t$ and $\sin^2 t$ yields $\frac{x^2}{4} = \cos^2 t$ and $\frac{y^2}{16} = \sin^2 t$. This shows $\cos^2 t + \sin^2 t = \frac{x^2}{4} + \frac{y^2}{16} = 1$, and as we suspected—the result is a vertical ellipse with vertices at $(0, \pm 4)$ and endpoints of the minor axis at $(\pm 2, 0)$.

Now try Exercises 13 through 16, Part b ▶

It's important to realize that a given curve can be represented parametrically in infinitely many ways. This flexibility sometimes enables us to simplify the given form, or to write a given polynomial form in an equivalent nonpolynomial form. The easiest way to write the function $y = f(x)$ in parametric form is $x = t$; $y = f(t)$, which is valid as long as t is in the domain of $f(t)$.

EXAMPLE 5 ▶ Writing an Equation in Terms of Various Parameters

Write the equation $y = 4(x-3)^2 + 1$ in three different parametric forms.

Solution ▶

- If we let $x = t$, we have $y = 4(t-3)^2 + 1$.
- Letting $x = t + 3$ simplifies the related equation for y , and we begin to see some of the advantages of using a parameter: $x = t + 3$; $y = 4t^2 + 1$.
- As a third alternative, we can let $x = \frac{1}{2} \tan t + 3$, which gives $x = \frac{1}{2} \tan t + 3$; $y = 4\left(\frac{1}{2} \tan t\right)^2 + 1 = \tan^2 t + 1$ or $y = \sec^2 t$.

✓ **B.** You've just learned how to write parametric equations in rectangular form

Now try Exercises 19 through 26 ▶

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C. Graphing Curves from the Cycloid Family

The **cycloids** are an important family of curves, and are used extensively to solve what are called **brachistochrone** applications. The name comes from the Greek *brachus*, meaning short, and *khoros*, meaning time, and deal with finding the path along which a weight will fall in the shortest time possible. Cycloids are an excellent example of why parametric equations are important, as it's very difficult to name them in rectangular form. Consider a point fixed to the circumference of a wheel as it rolls from left to right. If we trace the path of the point as the wheel rolls, the resulting curve is a cycloid. Figure 10.63 shows the location of the point every one-quarter turn.

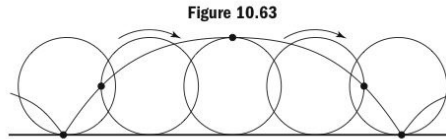


Figure 10.63

By superimposing a coordinate grid on the diagram in Figure 10.63, we can construct parametric equations that will produce the graph. This is done by developing equations for the location of a point $P(x, y)$ on the circumference of a circle with center (h, k) , as the circle rotates through angle t . After a rotation of t rad, the x -coordinate of $P(x, y)$ is $x = h - a$ (Figure 10.64), and the y -coordinate is $y = k - b$. Using a right triangle with the radius as the hypotenuse, we find $\sin t = \frac{a}{r}$ and $\cos t = \frac{b}{r}$, giving $a = r \sin t$ and $b = r \cos t$. Substituting into $x = h - a$ and $y = k - b$ yields $x = h - r \sin t$ and $y = k - r \cos t$. Since the circle has radius r , we know $k = r$ (the "height" of the center is constantly $k = r$). The arc length subtended by t is the same as the distance h (see Figure 10.65), meaning $h = rt$ (t in radians). Substituting rt for h and r for k in the equations $x = h - r \sin t$ and $y = k - r \cos t$, gives the equation of the cycloid in parametric form: $x = rt - r \sin t$ and $y = r - r \cos t$, sometimes written $x = r(t - \sin t)$ and $y = r(1 - \cos t)$.

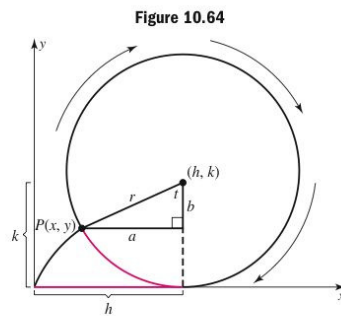


Figure 10.64

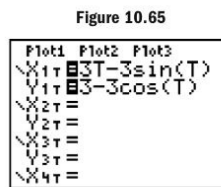


Figure 10.65

Most graphers have a parametric **MODE** that enables you to enter the equations for x and y separately, and graph the resulting points as a single curve. After pressing the **Y=** key (in parametric mode), the screen in Figure 10.65 comes into view using a TI-84 Plus, and we enter the equation of the cycloid formed by a circle of radius $r = 3$. To set the viewing window (including a frame), press **WINDOW** and set $Y_{min} = -1$ and Y_{max} at slightly more than 6 (since $r = 3$). Since the cycloid completes one cycle every $2\pi r$, we set X_{max} at $2\pi n$, where n is the number of cycles we'd like to see. In

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10-81

Section 10.7 Parametric Equations and Graphs

999

Figure 10.66

```

WINDOW
Tmin=0
Tmax=25.132741...
Tstep=.5235987...
Xmin=-3.141592...
Xmax=75.398223...
Xscl=9.4247779...
↓Ymin=-1
    
```

this case, we set it for four cycles $(2\pi)(3)(4) = 24\pi$ (Figure 10.66). With $r = 3$ we conveniently set Xscl at $3(2\pi) = 6\pi \approx 18.8$ to tick each cycle, and Xscl = $3\pi \approx 9.4$ to tick each half cycle (Figure 10.66). For parametric equations, we must also specify a range of values for t , which we set at Tmin = 0, Tmax = $8\pi \approx 25.1$ for the four cycles, and Tstep = $\frac{\pi}{6} \approx 0.52$ (Tstep controls the number of points plotted and joined to form the curve). The window settings and resulting graph are shown in Figure 10.67, which doesn't look much like a cycloid because the current settings do not produce a square viewing window. Using **ZOOM 5:ZSquare** (and changing Yscl) produces the graph shown in Figure 10.68, which looks much more like the cycloid we expected.

Figure 10.67

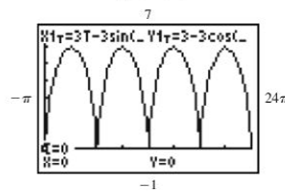
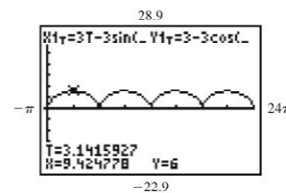


Figure 10.68

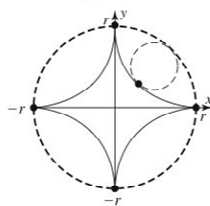


EXAMPLE 6 ▶ Using Technology to Graph a Cycloid

Use a graphing calculator to graph the curve defined by the equations $x = 3 \cos^3 t$ and $y = 3 \sin^3 t$, called a **hypocycloid with four cusps**.

Solution ▶ A hypocycloid is a curve traced out by the path of a point on the circumference of a circle as it rolls *inside* a larger circle of radius r (see Figure 10.69). Here $r = 3$ and we set Xmax and Ymax accordingly. Knowing ahead of time the hypocycloid will have four cusps, we set Tmax = $4(2\pi) \approx 25.13$ to show all four. The window settings used and the resulting graph are shown in Figures 10.70 and 10.71.

Figure 10.69



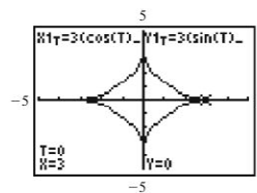
C. You've just learned how to graph curves from the cycloid family

Figure 10.70

```

WINDOW
Tmin=0
Tmax=25.132741...
Tstep=.5235987...
Xmin=-5
Xmax=5
Xscl=1
↓Ymin=-5
    
```

Figure 10.71



Now try Exercises 27 through 35 ▶

D. Common Applications of Parametric Equations

In Example 1 the parameter was simply the *real number* t , which enabled us to model the x - and y -values of an ordered pair (x, y) independently. In Examples 2 and 6, the parameter t represented an *angle*. Here we introduce yet another kind of parameter, that of *time* t .

A **projectile** is any object thrown, dropped, or projected in some way with no continuing source of propulsion. The parabolic path traced out by the projectile (assuming negligible air resistance) will be fully developed in Section 7.4. It is stated here in

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parametric terms. For the projectile's location $P(x, y)$ and any time t in seconds, the x -coordinate (horizontal distance from point of projection) is given by $x = v_0 t \cos \theta$, where v_0 is the initial velocity in feet per second and t is the time in seconds. The y -coordinate (vertical height) is $y = v_0 t \sin \theta - 16t^2$.

EXAMPLE 7 ▶ Using Parametric Equations in Projectile Applications

As part of a circus act, Karl the Human Cannonball is shot out of a specially designed cannon at an angle of 40° with an initial velocity of 120 ft/sec. Use a graphing calculator to graph the resulting parametric curve. Then use the graph to determine how high the Ring Master must place a circular ring for Karl to be shot through at the maximum height of his trajectory, and how far away the net must be placed to catch Karl.

Solution ▶ The information given leads to the equations $x = 120t \cos 40^\circ$ and $y = 120t \sin 40^\circ - 16t^2$. Enter these equations on the **Y=** screen of your calculator, remembering to reset the **MODE** to degrees (circus clowns may not know or understand radians). To set the window size, we can use trial and error, or estimate using $\theta = 45^\circ$ (instead of 40°) and an estimate for t (the time that Karl will stay aloft). With $t = 6$ we get estimates of $x = 120(6)\left(\frac{\sqrt{2}}{2}\right) = 360\sqrt{2}$ for the horizontal distance. To find a range for y , use $t = 3$ since the maximum height of the parabolic path will occur halfway through the flight. This gives an estimate of $120(3)\left(\frac{\sqrt{2}}{2}\right) - 16(9) = 180\sqrt{2} - 144$ for y . The results are shown in Figures 10.72 and 10.73. Using the **TRACE** feature or **2nd** **GRAPH** (**TABLE**) feature, we find the center of the net used to catch Karl should be set at a distance of about 450 ft from the cannon, and the ring should be located 220 ft from the cannon at a height of about 93 ft.

Figure 10.72

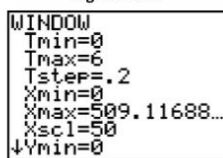
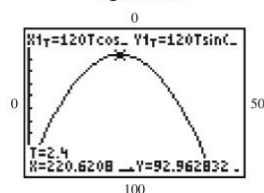


Figure 10.73



Now try Exercises 46 through 49 ▶

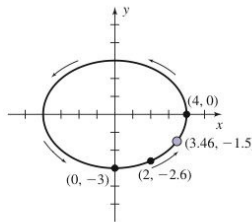
It is well known that planets orbit the Sun in elliptical paths. While we're able to model their orbits in both rectangular and polar form, neither of these forms can give a true picture of the *direction they travel*. This gives parametric forms a great advantage, in that they can model the shape of the orbit, *while also indicating the direction of travel*. We illustrate in Example 8 using a "planet" with a very simple orbit.

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EXAMPLE 8 ▶ **Modeling Elliptical Orbits Parametrically**

The elliptical orbit of a certain planet is defined parametrically as $x = 4 \sin t$ and $y = -3 \cos t$. Graph the orbit and verify that for increasing values of t , the planet orbits in a counterclockwise direction.

Solution ▶ Eliminating the parameter as in Example 4, we obtain the equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$, or the equation of an ellipse with center at $(0, 0)$, major axis of length 8, and minor axis of length 6. The path of the planet is traced out by the ordered pairs (x, y) generated by the parametric equations, shown in the table for $t \in [0, \pi]$. Starting at $t = 0$, $P(x, y)$ begins at $(0, -3)$ with x and y both increasing until $t = \frac{\pi}{2}$. Then from $t = \frac{\pi}{2}$ to $t = \pi$, y continues to increase as x decreases, indicating a counterclockwise orbit in this case. The orbit is illustrated in the figure.



t	$x = 4 \sin t$	$y = -3 \cos t$
0	0	-3
$\frac{\pi}{6}$	2	-2.6
$\frac{\pi}{3}$	3.46	-1.5
$\frac{\pi}{2}$	4	0
$\frac{2\pi}{3}$	3.46	1.5
$\frac{5\pi}{6}$	2	2.6
π	0	3

Now try Exercises 50 and 51 ▶

Finally, you may recall from your previous work with linear 3×3 systems, that a dependent system occurs when one of the three equations is a linear combination of the other two. The result is a system with more variables than equations, with solutions expressed in terms of a parameter, or in *parametric form*. These solutions can be explored on a graphing calculator using ordered triples of the form $(t, f(t), g(t))$, where $Y_1 = f(t)$ and $Y_2 = g(t)$ (see Exercises 52 through 55). For more information, see the *Calculator Exploration and Discovery* feature on page 1012.

✓ **D.** You've just learned how to solve applications involving parametric equations

TECHNOLOGY HIGHLIGHT

Exploring Parametric Graphs

Most graphing calculators have features that make it easy (and fun) to explore parametric equations. For example, the TI-84 Plus can use a circular cursor to trace the path of the plotted points, as they are generated by the equations. This can be used to illustrate the path of a projectile, the distance of a runner, or the orbit of a planet. Operations can also be applied to the parameter T to give the effect of "speed" (the points from one set of equations are plotted faster than the points of a second set). To help illustrate their use, consider again the simple, elliptical orbit of a planet in Example 8. Physics tells us the closer a planet is to the Sun, the faster its orbit. In fact, the orbital speed of Mercury is about twice that of Mars and about 10 times as fast as the dwarf planet Pluto (29.8, 15, and 2.9 mi/sec,

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respectively). With this information, we can explore a number of interesting questions. On the Y= screen, let the orbits of Planet 1 and Planet 2 be modeled parametrically by the equations shown in Figure 10.74. Since the orbit of Planet 1 is "smaller" (closer to the Sun), we have T -values growing at a rate that is *four times as fast* as for Planet 2. Notice to the far left of X_{1T} , there is a symbol that looks like an old key " -0 ." By moving the cursor to the far left of the equation, you can change how the graph will look by repeatedly pressing ENTER . With this symbol in view, the calculator will trace out the curve with a circular cursor, which in this case represents the planets as they orbit (be sure you are in simultaneous MODE). Setting the window as in Figure 10.75 and pressing GRAPH produces Figure 10.76, which displays their elliptical paths as they race around the Sun. Notice the inner planet has already completed one orbit while the outer planet has just completed one-fourth of an orbit.

Figure 10.74

```

P1ot1 P1ot2 P1ot3
-0X1T 4sin(4T)
Y1T -3cos(4T)
-0X2T 8sin(T)
Y2T -5cos(T)
X3T =
Y3T =
X4T =

```

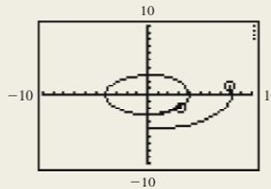
Figure 10.75

```

WINDOW
Tmin=0
Tmax=25
Tstep=.1
Xmin=-10
Xmax=10
Ysc1=1
Ymin=-10

```

Figure 10.76



Exercise 1: Verify that the inner planet completes four orbits for every single orbit of the outer planet.

Exercise 2: Suppose that due to some cosmic interference, the orbit of the faster planet begins to decay at a rate of $T^{0.84}$ (replace T with $T^{0.84}$ in both equations for the inner planet). By observation, about how many orbits did the inner planet make for the first revolution of the outer planet? What is the ratio of orbits for the next complete orbit of the outer planet?



10.7 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

- When the coordinates of a point (x, y) are generated independently using $x = f(t)$ and $y = g(t)$, t is called a(n) _____.
- The equations $x = f(t)$ and $y = g(t)$ used to generate the ordered pairs (x, y) are called _____ equations.
- Parametric equations can both graph a curve and indicate the _____ traveled by a point on the curve.
- To write parametric equations in rectangular form, we must _____ the parameter to write a single equation.
- Discuss the connection between solutions to dependent systems and the parametric equations studied in this section.
- In your own words, explain and illustrate the process used to develop the equation of a cycloid. Illustrate with a specific example.

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► **DEVELOPING YOUR SKILLS**

For Exercises 7 through 18, (a) graph the curves defined by the parametric equations using the specified interval and identify the graph (if possible) and (b) eliminate the parameter (Exercises 7 to 16 only) and write the corresponding rectangular form.

$$7. x = t + 2; t \in [-3, 3]$$

$$y = t^2 - 1$$

$$8. x = t - 3; t \in [-5, 5]$$

$$y = 2 - 0.5t^2$$

$$9. x = (2 - t)^2; t \in [0, 5]$$

$$y = (t - 3)^2$$

$$10. x = t^3 - 3; t \in [-2, 2.5]$$

$$y = t^2 + 1$$

$$11. x = \frac{5}{t}, t \neq 0; t \in [-3.5, 3.5]$$

$$y = t^2$$

$$12. x = \frac{t^3}{10}; t \in [-5, 5]$$

$$y = |t|$$

$$13. x = 4 \cos t; t \in [0, 2\pi]$$

$$y = 3 \sin t$$

$$14. x = 2 \sin t; t \in [0, 2\pi]$$

$$y = -3 \cos t$$

$$15. x = 4 \sin(2t); t \in [0, 2\pi]$$

$$y = 6 \cos t$$

$$16. x = 4 \cos(2t); t \in \left[\frac{\pi}{2}, \frac{3\pi}{2} \right]$$

$$y = 6 \sin t$$

$$17. x = \frac{-3}{\tan t}; t \in (0, \pi)$$

$$y = 5 \sin(2t)$$

$$18. x = \tan^2 t; t \neq \frac{\pi}{2}, t \in [0, \pi]$$

$$y = 3 \cos t$$

Write each function in three different parametric forms by altering the parameter. For Exercises 19–22 use at least one trigonometric form, restricting the domain as needed.

$$19. y = 3x - 2 \qquad 20. y = 0.5x + 6$$

$$21. y = (x + 3)^2 + 1 \qquad 22. y = 2(x - 5)^2 - 1$$

$$23. y = \tan^2(x - 2) + 1 \qquad 24. y = \sin(2x - 1)$$

25. Use a graphing calculator or computer to verify that the parametric equations from Example 5 all produce the same graph.

26. Use a graphing calculator or computer to verify that your parametric equations from Exercise 21 all produce the same graph.

The curves defined by the following parametric equations are from the cycloid family. (a) Use a graphing calculator or computer to draw the graph and (b) use the graph to approximate all x - and y -intercepts, and maximum and minimum values to one decimal place.

$$27. x = 8 \cos t + 2 \cos(4t), y = 8 \sin t - 2 \sin(4t),$$

hypocycloid (5-cusp)

$$28. x = 8 \cos t + 4 \cos(2t), y = 8 \sin t - 4 \sin(2t),$$

hypocycloid (3-cusp)

$$29. x = \frac{2}{\tan t}, y = 8 \sin t \cos t, \text{ serpentine curve}$$

$$30. x = 8 \sin^2 t, y = \frac{8 \sin^3 t}{\cos t}, \text{ cissoid of Diocles}$$

$$31. x = 2(\cos t + t \sin t), y = 2(\sin t - t \cos t),$$

involute of a circle

$$32. 4x = (16 - 36)\cos^3 t, 6y = (16 - 36)\sin^3 t,$$

evolute of an ellipse

$$33. x = 3t - \sin t, y = 3 - \cos t, \text{ curtate cycloid}$$

$$34. x = t - 3 \sin t, y = 1 - 3 \cos t, \text{ prolate cycloid}$$

$$35. x = 2[3 \cos t - \cos(3t)], y = 2[3 \sin t - \sin(3t)],$$

nephroid



Use a graphing calculator or computer to draw the following parametrically defined graphs, called Lissajous figures (Exercise 37 is a scaled version of the initial example from this section). Then find the dimensions of the rectangle necessary to frame the figure and state the number of times the graph crosses itself.

$$36. x = 6 \sin(3t) \qquad 37. x = 6 \sin(2t)$$

$$y = 8 \cos t \qquad y = 8 \cos t$$

$$38. x = 8 \sin(4t) \qquad 39. x = 5 \sin(7t)$$

$$y = 10 \cos t \qquad y = 7 \cos(4t)$$

$$40. x = 8 \sin(4t) \qquad 41. x = 10 \sin(1.5t)$$

$$y = 10 \cos(3t) \qquad y = 10 \cos(2.5t)$$

42. Use a graphing calculator to experiment with parametric equations of the form $x = A \sin(mt)$ and $y = B \cos(nt)$. Try different values of A , B , m , and n , then discuss their effect on the Lissajous figures.

43. Use a graphing calculator to experiment with parametric equations of the form $x = \frac{a}{\tan t}$ and $y = b \sin t \cos t$. Try different values of a and b , then discuss their effect on the resulting graph, called a serpentine curve. Also see Exercise 29.

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▶ WORKING WITH FORMULAS

44. The Folium of Descartes:

$$x(t) = \frac{3kt}{1+t^3}; y(t) = \frac{3kt^2}{1+t^3}$$

The Folium of Descartes is a parametric curve developed by Descartes in order to test the ability of Fermat to find its maximum and minimum values.


- Graph the curve on a graphing calculator with $k = 1$ using a reduced window (**ZOOM** 4), with $T_{\min} = -6$, $T_{\max} = 6$, and $T_{\text{step}} = 0.1$. Locate the coordinates of the tip of the folium (the loop).
- This graph actually has a discontinuity (a break in the graph). At what value of t does this occur?
- Experiment with different values of k and generalize its effect on the basic graph.

45. The Witch of Agnesi: $x(t) = 2kt; y(t) = \frac{2k}{1+t^2}$

The Witch of Agnesi is a parametric curve named by Maria Agnesi in 1748. Some believe she confused the Italian word for *witch* (*versiera*), with a similar word that meant *free to move*. In any case, the name stuck. The curve can also be stated in trigonometric form: $x(t) = 2k \cot t$ and $y = 2k \sin^2 t$.

- Graph the curve with $k = 1$ on a calculator or computer on a reduced window (**ZOOM** 4) using both of the forms shown with $T_{\min} = -6$, $T_{\max} = 6$, and $T_{\text{step}} = 0.1$. Try to determine the maximum value.
- Explain why the x -axis is a horizontal asymptote.
- Experiment with different values of k and generalize its effect on the basic graph.

▶ APPLICATIONS

 Model each application using parametric equations, then solve using the **GRAPH** and **TRACE** features of a graphing calculator.

- 46. Archery competition:** At an archery contest, a large circular target 5 ft in diameter is laid flat on the ground with the bull's-eye exactly 180 yd (540 ft) away from the archers. Marion draws her bow and shoots an arrow at an angle of 25° above horizontal with an initial velocity of 150 ft/sec (assume the archers are standing in a depression and the arrow is shot from ground level). (a) What was the maximum height of the arrow? (b) Does the arrow hit the target? (c) What is the distance between Marion's arrow and the bull's-eye after the arrow hits?



- 47. Football competition:** As part of their contribution to charity, a group of college quarterbacks participate in a contest. The object is to throw a football through a hoop whose center is 30 ft high and 25 yd (75 ft) away, trying to hit a stationary (circular) target laid on the ground with the center

56 yd (168 ft) away. The hoop and target both have a diameter of 4 ft. On his turn, Lance throws the football at an angle of 36° with an initial velocity of 75 ft/sec. (a) Does the football make it through the hoop? (b) Does the ball hit the target? (c) What is the approximate distance between the football and the center of the target when the ball hits the ground?

- 48. Walk-off home run:** It's the bottom of the ninth, two outs, the count is full, and the bases are loaded with the opposing team ahead 5 to 2. The home team has Heavy Harley, their best hitter at the plate; the opposition has Raymond the Rocket on the mound. Here's the pitch . . . it's hit . . . a long fly ball to left-center field! If the ball left the bat at an angle of 30° with an initial velocity of 112 ft/sec, will it clear the home run fence, 9 ft high and 320 ft away?



- 49. Last-second win:** It's fourth-and-long, late in the fourth quarter of the homecoming football game, with the home team trailing 29 to 27. The coach elects to kick a field goal, even though the goal posts are 50 yd (150 ft) away from the spot of the kick. If the ball leaves the kicker's foot at an angle

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10-87

of 29° with an initial velocity of 80 ft/sec, and the kick is "true," will the home team win (does the ball clear the 10-ft high cross bar)?



50. Particle motion: The motion of a particle is modeled by the parametric equations

$$\begin{cases} x = 5t - 2t^2 \\ y = 3t - 2 \end{cases} \text{ . Between } t = 0 \text{ and } t = 1, \text{ is the particle moving to the right or to the left? Is the particle moving upward or downward?}$$

51. Electron motion: The motion of an electron as it orbits the nucleus is modeled by the parametric equations $\begin{cases} x = 6 \cos t \\ y = 2 \sin t \end{cases}$ with t in radians. Between $t = 2$ and $t = 3$, is the electron moving to the right or to the left? Is the electron moving upward or downward?

Systems applications: Solve the following systems using elimination. If the system is dependent, write the general solution in parametric form and use a calculator to generate several solutions.

$$52. \begin{cases} 2x - y + 3z = -3 \\ 3x + 2y - z = 4 \\ 8x + 3y + z = 5 \end{cases} \quad 53. \begin{cases} x - 5y + z = 3 \\ 5x + y - 7z = -9 \\ 2x + 3y - 4z = -6 \end{cases}$$

$$54. \begin{cases} -5x - 3z = -1 \\ x + 2y - 2z = -3 \\ -2x + 6y - 9z = -10 \end{cases}$$

$$55. \begin{cases} x + y - 5z = -4 \\ 2y - 3z = -1 \\ x - 3y + z = -3 \end{cases}$$

► **EXTENDING THE CONCEPT**

58. What is the difference between an *epicycloid*, a *hypercyloid*, and a *hypocycloid*? Do a word study on the prefixes *epi-*, *hyper-*, and *hypo-*, and see

Section 10.7 Parametric Equations and Graphs

1005

56. Regressions and parameters:

Draw a scatter-plot of the data given in the table. Note that connecting the points with a smooth curve will not result in a function, so a standard regression cannot be run on the data. Now consider the x -values alone—what do you notice? Find a sinusoidal model for the x -values, using $T = 0, 1, 2, 3, \dots, 8$. Use the same inputs to run some form of regression on the y -values, then use the results to form the "best-fit" parametric equations for this data (use L1 for T, L2 for the x -values, and L3 for the y -values). With your calculator in parametric **MODE**, enter the equations as X_{IT} and Y_{IT} , then graph these along with the scatterplot (L2, L3) to see the finished result. Use the **TABLE** feature of your calculator to comment on the accuracy of the model.

x	y
0	0
$\sqrt{2}$	0.25
2	2
$\sqrt{2}$	6.75
0	16
$-\sqrt{2}$	31.25
-2	54
$-\sqrt{2}$	85.75
0	128

57. Regressions and parameters:

Draw a scatter-plot of the data given in the table, and connect the points with a smooth curve. The result is a function, but no standard regression seems to give an accurate model. The x -values alone are actually generated by an exponential function. Run a regression on these values using $T = 0, 1, 2, 3, \dots, 8$ as inputs to find the exponential model. Then use the same inputs to run some form of regression on the y -values and use the results to form the "best-fit" parametric equations for this data (use L1 for T, L2 for the x -values, and L3 for the y -values). With your calculator in parametric **MODE**, enter the equations as X_{IT} and Y_{IT} , then graph these along with the scatterplot (L2, L3) to see the finished result. Use the **TABLE** feature of your calculator to comment on the accuracy of the model.

x	y
1	0
1.2247	-1.75
1.5	-3
1.8371	-3.75
2.25	-4
2.7557	-3.75
3.375	-3
4.1335	-1.75
5.0625	0

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1104

Coburn: Algebra and
Trigonometry, Second
Edition

10. Analytic Geometry and
the Conic Sections

10.7: Parametric Equations
and Graphs

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1006

CHAPTER 10 Analytic Geometry and the Conic Sections

10-88

59. The motion of a particle in a certain medium is modeled by the parametric equations $\begin{cases} x = 6 \sin(4t) \\ y = 8 \cos t \end{cases}$.

Initially, use only the **2nd** **GRAPH** **(TABLE)** feature of your calculator (not the graph) to name the intervals for which the particle is moving (a) to the left and upward and (b) to the left and downward. Answer to the nearest tenth (set $\Delta Tbl = 0.1$). Is it *possible* for this particle to collide with another particle in this medium whose

movement is modeled by $\begin{cases} x = 3 \cos t + 7 \\ y = 2 \sin t + 2 \end{cases}$? Discuss why or why not.

60. Write the function $y = \frac{1}{2}(x + 3)^2 - 1$ in parametric form using the substitution $x = 2 \cos t - 3$ and the appropriate double-angle identity. Is the result equivalent to the original function? Why or why not?

► MAINTAINING YOUR SKILLS

61. **(1.1)** The price of a popular video game is reduced by 20% and is selling for \$39.96. By what percentage must the sale price be increased to return the item to its original price?
62. **(5.2)** When the tip of the antenna atop the Eiffel Tower is viewed at a distance of 265 ft from its base, the angle of elevation is 76° . Is the Eiffel Tower taller or shorter than the Chrysler Building (New York City) at 1046 ft?
63. **(3.4)** Graph $f(x) = x^3 + 2x^2 - 5x - 6$ using information about end behavior, y-intercept, x-intercept(s), and midinterval points:
64. **(6.6)** The maximum height a projectile will attain depends on the angle it is projected and its initial velocity. This phenomena is modeled by the function $H = \frac{v^2 \sin^2 \theta}{64}$, where v is the initial velocity (in feet/sec) of the projectile and θ is the angle of projection. Find the angle of projection if the projectile attained a maximum height of 151 ft, and the initial velocity was 120 ft/sec.

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SUMMARY AND CONCEPT REVIEW

SECTION 10.1 A Brief Introduction to Analytical Geometry

KEY CONCEPTS

- The midpoint and distance formulas play an important role in the study of analytical geometry:

$$\text{midpoint: } (x, y) = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2} \right) \quad \text{distance: } d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- The perpendicular distance from a point to a line is the length of a line segment perpendicular to a given line with the given point and the point of intersection as endpoints.
- Using these tools, we can verify or construct relationships between points, lines, and curves in the plane; verify properties of geometric figures; prove theorems from Euclidean geometry; and construct relationships that define the conic sections.

EXERCISES

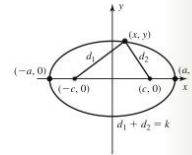
- Verify the closed figure with vertices $(-3, -4)$, $(-5, 4)$, $(3, 6)$, and $(5, -2)$ is a square.
- Find the equation of the circle that circumscribes the square in Exercise 1.
- A theorem from Euclidean geometry states: *If any two points are equidistant from the endpoints of a line segment, they are on the perpendicular bisector of the segment.* Determine if the line through $(-3, 6)$ and $(6, -9)$ is a perpendicular bisector of the segment through $(-5, -2)$ and $(5, 4)$.
- Four points are given below. Verify that the distance from each point to the line $y = -1$ is the same as the distance from the given point to the fixed point $(0, 1)$: $(-6, 9)$, $(-2, 1)$, $(4, 4)$, and $(8, 16)$.

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SECTION 10.2 The Circle and the Ellipse

KEY CONCEPTS

- The equation of a circle centered at (h, k) with radius r is $(x - h)^2 + (y - k)^2 = r^2$.
- Dividing both sides by r^2 , we obtain the standard form $\frac{(x - h)^2}{r^2} + \frac{(y - k)^2}{r^2} = 1$, showing the horizontal and vertical distance from center to graph is r .
- The equation of an ellipse in standard form is $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$. The center of the ellipse is (h, k) , with horizontal distance a and vertical distance b from center to graph.
- Given two fixed points f_1 and f_2 in a plane (called the foci), an ellipse is the set of all points (x, y) such that the distance from the first focus to (x, y) , plus the distance from the second focus to (x, y) , remains constant.
- For an ellipse, the distance a from center to vertex is *greater than* the distance c from center to one focus.
- To find the foci of an ellipse: $a^2 = b^2 + c^2$ (since $a > c$).



EXERCISES

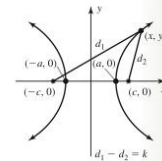
Sketch the graph of each equation in Exercises 5 through 9.

- $x^2 + y^2 = 16$
- $x^2 + 4y^2 = 36$
- $9x^2 + y^2 - 18x - 27 = 0$
- $x^2 + y^2 + 6x + 4y + 12 = 0$
- $\frac{(x + 3)^2}{16} + \frac{(y - 2)^2}{9} = 1$
- Find the equation of the ellipse with minor axis of length 6 and foci at $(-4, 0)$ and $(4, 0)$.
- Find the equation of the ellipse with vertices at (a) $(-13, 0)$ and $(13, 0)$, foci at $(-12, 0)$ and $(12, 0)$; (b) foci at $(0, -16)$ and $(0, 16)$, major axis: 40 units.
- Write the equation in standard form and sketch the graph, noting all of the characteristic features of the ellipse. $4x^2 + 25y^2 - 16x - 50y - 59 = 0$

SECTION 10.3 The Hyperbola

KEY CONCEPTS

- The equation of a *horizontal* hyperbola in standard form is $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$. The center of the hyperbola is (h, k) with horizontal distance a from center to vertices and vertical distance b from center to the midpoint of one side of the central rectangle.
- Given two fixed points f_1 and f_2 in a plane (called the foci), a hyperbola is the set of all points (x, y) such that the distance from the first focus to point (x, y) , less the distance from the second focus to (x, y) , remains constant.
- For a hyperbola, the distance from center to one of the vertices is *less than* the distance from center to one focus.
- To find the foci of a hyperbola: $c^2 = a^2 + b^2$ (since $c > a$).



EXERCISES

Sketch the graph of each equation, indicating the center, vertices, and asymptotes. For Exercise 18, also give the equation of the hyperbola in standard form.

- $4y^2 - 25x^2 = 100$
- $\frac{(y - 3)^2}{16} - \frac{(x + 2)^2}{9} = 1$
- $\frac{(x + 2)^2}{9} - \frac{(y - 1)^2}{4} = 1$

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1008

CHAPTER 10 Analytic Geometry and the Conic Sections

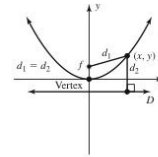
10-90

16. $9y^2 - x^2 - 18y - 72 = 0$ 17. $x^2 - 4y^2 - 12x - 8y + 16 = 0$
18. vertices at $(-3, 0)$ and $(3, 0)$, asymptotes of $y = \pm \frac{4}{3}x$
19. Find the equation of the hyperbola with (a) vertices at $(\pm 15, 0)$, foci at $(\pm 17, 0)$, and (b) foci at $(0, \pm 5)$ with vertical dimension of central rectangle 8 units.
20. Write the equation in standard form and sketch the graph, noting all of the characteristic features of the hyperbola. $4x^2 - 9y^2 - 40x + 36y + 28 = 0$

SECTION 10.4 The Analytic Parabola

KEY CONCEPTS

- Horizontal parabolas have equations of the form $x = ay^2 + by + c$; $a \neq 0$.
- A horizontal parabola will open to the right if $a > 0$, and to the left if $a < 0$. The axis of symmetry is $y = \frac{-b}{2a}$, with the vertex (h, k) found by evaluating at $y = \frac{-b}{2a}$ or by completing the square and writing the equation in shifted form: $x = a(y - k)^2 + h$.
- Given a fixed point f (called the focus) and fixed line D in the plane, a parabola is the set of all points (x, y) such that the distance from f to (x, y) is equal to the distance from (x, y) to line D .
- The equation $x^2 = 4py$ describes a vertical parabola, opening upward if $p > 0$, and opening downward if $p < 0$.
- The equation $y^2 = 4px$ describes a horizontal parabola, opening to the right if $p > 0$, and opening to the left if $p < 0$.
- The focal chord of a parabola is a line segment that contains the focus and is parallel the directrix, with its endpoints on the graph. It has a total length of $|4p|$, meaning the distance from the focus to a point of the graph is $|2p|$. It is commonly used to assist in drawing a graph of the parabola.



EXERCISES

For Exercises 21 and 22, find the vertex and x - and y -intercepts if they exist. Then sketch the graph using symmetry and a few points or by completing the square and shifting a parent function.

21. $x = y^2 - 4$ 22. $x = y^2 + y - 6$

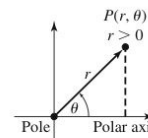
For Exercises 23 and 24, find the vertex, focus, and directrix for each parabola. Then sketch the graph using this information and the focal chord. Also graph the directrix.

23. $x^2 = -20y$ 24. $x^2 - 8x - 8y + 16 = 0$

SECTION 10.5 Polar Coordinates, Equations, and Graphs

KEY CONCEPTS

- In polar coordinates, the location of a point in the plane is denoted (r, θ) , where r is the distance to the point from the origin or *pole*, and θ is the angle between a stipulated polar axis and a ray containing P .
- In the polar coordinate system, the location (r, θ) of a point is not unique for two reasons: (1) the angles θ and $\theta + 2\theta n$ are coterminal (n an integer), and (2) r may be negative.
- The point $P(r, \theta)$ can be converted to $P(x, y)$ in rectangular coordinates where $x = r \cos \theta$ and $y = r \sin \theta$.



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- The point $P(x, y)$ in rectangular coordinates can be converted to $P(r, \theta)$ in polar coordinates, where $r = \sqrt{x^2 + y^2}$ and $\theta_r = \tan^{-1}\left(\frac{y}{x}\right)$.
- To sketch a polar graph, we view the length r as being along the second hand of a clock, ticking in a counterclockwise direction. Each "tick" is $\frac{\pi}{12}$ rad or 15° . For each tick we locate a point on the radius and plot it on the face of the clock before going on.
- For graphing, we also apply an " r -value" analysis, which looks where r is increasing, decreasing, zero, maximized, and/or minimized.
- If the polar equation is given in terms of sines, the graph will be symmetric to $\theta = \frac{\pi}{2}$.
- If the polar equation is given in terms of cosines, the graph will be symmetric to the polar axis.
- The graphs of several common polar equations are given in Appendix V.

EXERCISES

Sketch using an r -value analysis (include a table), symmetry, and any convenient points.

25. $r = 5 \sin \theta$ 26. $r = 4 + 4 \cos \theta$ 27. $r = 2 + 4 \cos \theta$ 28. $r = 8 \sin(2\theta)$

SECTION 10.6 More on the Conic Sections: Rotation of Axes and Polar Form**KEY CONCEPTS**

- Using a rotation, the conic equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ in the xy -plane can be transformed into $aX^2 + cY^2 + dX + eY + f = 0$ in the XY -plane, in which the mixed xy -term is absent.
- The required angle of rotation β is found using $\tan(2\beta) = \frac{B}{A - C}$; $0 < 2\beta < 180^\circ$.
- The change in coordinates from the xy -plane to the XY -plane is accomplished using the rotation formulas:

$$x = X \cos \beta - Y \sin \beta \quad y = X \sin \beta + Y \cos \beta$$
- In the process of this conversion, certain quantities, called invariants, remain unchanged and can be used to check that the conversion was correctly performed. These invariants are (1) $F = f$, (2) $A + C = a + c$, and (3) $b^2 - 4AC = b^2 - 4ac$.
- The invariants $B^2 - 4AC = b^2 - 4ac$ are called discriminants and can be used to classify the type of graph the equation will give, except in degenerate cases:
 - If $B^2 - 4AC = 0$, the equation is that of a parabola.
 - If $B^2 - 4AC < 0$, the equation is that of a circle or an ellipse.
 - If $B^2 - 4AC > 0$, the equation is that of a hyperbola.
- All conics (not only the parabola) can be stated in terms of a focus/directrix definition. This is done using the concept of eccentricity, symbolized by the letter e .
- If F is a fixed point and \mathcal{L} a fixed line in the plane with the point D on \mathcal{L} , the set of all points P such that $\frac{FP}{DP} = e$ (e a constant) is the graph of a conic section. If $e = 1$, the graph is a parabola. If $0 < e < 1$, the graph is an ellipse. If $e > 1$, the graph is a hyperbola.
- Given a conic section with eccentricity e , one focus at the pole of the $r\theta$ -plane, and directrix \mathcal{L} located d units from this focus, then the polar equations $r = \frac{de}{1 \pm e \cos \theta}$ and $r = \frac{de}{1 \pm e \sin \theta}$ represent one of the conic sections as determined by the value of e .

EXERCISES

For the given conics in the xy -plane, use a rotation of axes to find the corresponding equation in the XY -plane, then sketch its graph.

29. $2x^2 - 4xy + 2y^2 - 8\sqrt{2}y - 24 = 0$ 30. $x^2 + 6\sqrt{3}xy + 7y^2 - 160 = 0$

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For the conic equations given, determine if the equation represents a parabola, ellipse, or hyperbola. Then describe and sketch the graphs using polar graph paper.

31. $r = \frac{9}{3 - 2 \cos \theta}$

32. $r = \frac{8}{4 - 6 \cos \theta}$

33. $r = \frac{4}{3 + 3 \sin \theta}$

34. Mars has a perihelion of 128.4 million miles and an aphelion of 154.9 million miles. Use this information to find a polar equation that models the elliptical orbit, then find the length of the focal chord.

SECTION 10.7 Parametric Equations and Graphs

KEY CONCEPTS

- If we consider the set of points $P(x, y)$ such that the x -values are generated by $f(t)$ and the y -values are generated by $g(t)$ (assuming f and g are both defined on an interval of the domain), the equations $x = f(t)$ and $y = g(t)$ are called parametric equations, with parameter t .
- Parametric equations can be converted to rectangular form by eliminating the parameter. This can sometimes be done by solving for t in one equation and substituting in the other, or by using trigonometric forms.
- A function can be written in parametric form many different ways, by altering the parameter or using trigonometric identities.
- The cycloids are an important family of curves, with equations $x = r(t - \sin t)$ and $y = r(1 - \cos t)$.
- The solutions to dependent systems of equations are often expression in parametric form, with the points $P(x, y)$ given by the parametric equations generating solutions to the system.

EXERCISES

Graph the curves defined by the parametric equations over the specified interval and identify the graph. Then eliminate the parameter and write the corresponding rectangular form.

35. $x = t - 4; t \in [-3, 3];$ 36. $x = (2 - t)^2; t \in [0, 5];$ 37. $x = -3 \sin t; t \in [0, 2\pi];$
 $y = -2t^2 + 3$ $y = (t - 3)^2$ $y = 4 \cos t$

38. Write the function in three different forms by altering the parameter: $y = 2(x - 5)^2 - 1$

39. Use a graphing calculator to graph the Lissajous figure indicated, then state the size of the rectangle needed to frame it: $x = 4 \sin(5t); y = 8 \cos t$

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MIXED REVIEW

For Exercises 1 through 16, graph the conic section and locate the center, vertices, directrix, foci, focal chords, asymptotes, and other important features as these apply to a particular equation and conic.

1. $9x^2 + 9y^2 = 54$

2. $16x^2 + 25y^2 = 400$

3. $9y^2 - 25x^2 = 225$

4. $\frac{(x-3)^2}{9} + \frac{(y+1)^2}{25} = 1$

5. $4(x-1)^2 - 36(y+2)^2 = 144$

6. $16(x+2)^2 + 4(y-1)^2 = 64$

7. $y = -2x^2 - 10x + 15$

8. $x = -y^2 - 8y - 11$

9. $x = y^2 + 2y + 3$

10. $x = (y+2)^2 - 3$

11. $x^2 - 8x - 8y + 16 = 0$

12. $x^2 = -24y$

13. $4x^2 - 25y^2 - 24x + 150y - 289 = 0$

14. $4x^2 + 16y^2 - 12x - 48y - 19 = 0$

15. $49(x+2)^2 + (y-3)^2 = 49$

16. $x^2 + y^2 - 8x + 12y + 16 = 0$

17. Graph the curve defined by the parametric equations given, using the interval $t \in [0, 10]$. Then identify the graph: $x = (t-2)^2$, $y = (t-4)^2$

18. Plot the polar coordinates given, then convert to rectangular coordinates.

a. $\left(3.5, \frac{2\pi}{3}\right)$ b. $\left(-4, \frac{5\pi}{4}\right)$

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10-93

Practice Test

1011

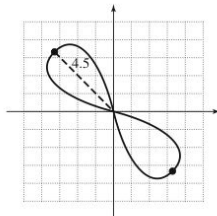
19. Solve using elimination:

a. $\begin{cases} 4x^2 - y^2 = -9 \\ x^2 + 3y^2 = 79 \end{cases}$ b. $\begin{cases} 4x^2 + 9y^2 = 36 \\ x^2 + 3y = 6 \end{cases}$

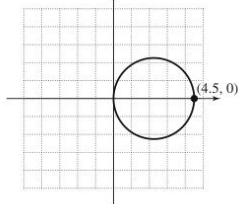
20. Match each equation to its corresponding graph. Justify each response.

- (i) $r = 3.5 + \cos \theta$
 (ii) $r^2 = 20.25 \sin(-2\theta)$
 (iii) $r = 4.5 \cos \theta$

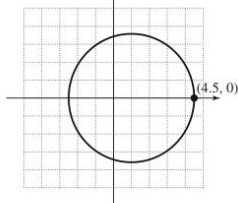
a.



b.



c.



21. A go-cart travels around an elliptical track with a 100-m major axis that is horizontal. The minor axis measures 60 m. Write an equation model for the track in parametric form.

22. Except for small variations, a planet's orbit around the Sun is elliptical, with the Sun at one focus. The *perihelion* or minimum distance from the planet Mercury to the Sun is about 46 million kilometers. Its *aphelion* or maximum distance from the Sun is approximately 70 million kilometers. Use this information to find the length of the major and minor axes, then determine the equation model for the orbit of Mercury in the standard form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

23. The orbit of a comet can also be modeled by one of the conic sections, with the Sun at one focus. Assuming the equations given model a comet's path, (1) determine if the path is circular, elliptic, hyperbolic, or parabolic; and (2) determine the closest distance the comet will come to the Sun (in millions of miles).

a. $r = \frac{84}{100 + 70 \cos \theta}$ b. $r = \frac{31}{5 - 5 \sin \theta}$

24. In the design of their corporate headquarters, Centurion Computing includes a seven-leaf rose in a large foyer, with a fountain in the center. Each of the leaves is 5 m long (when measured from the center of the fountain), and will hold flower beds for carefully chosen perennials. The rose is to be symmetric to a vertical axis, with the leaf bisected by $\theta = \frac{\pi}{2}$ pointing directly to the elevators. Find the equation of the rose in polar form.

25. The hyperbola defined by $\frac{X^2}{80^2} - \frac{Y^2}{400^2} = 1$ in the XY -plane is rotated clockwise by 45° . What is the corresponding equation in the xy -plane?

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PRACTICE TEST

By inspection only (no graphing), match each equation to its correct description.

1. $x^2 + y^2 - 6x + 4y + 9 = 0$ _____
 2. $4y^2 + x^2 - 4x + 8y + 20 = 0$ _____
 3. $x^2 - 4y^2 - 4x + 12y + 20 = 0$ _____
 4. $y - x^2 - 4x + 20 = 0$ _____
- a. Parabola b. Hyperbola c. Circle d. Ellipse

Identify and then graph each of the following conic sections. State the center, vertices, foci, asymptotes, and other important points when applicable.

5. $x^2 + y^2 - 4x + 10y + 20 = 0$
6. $25(x + 2)^2 + 4(y - 1)^2 = 100$
7. $r = \frac{10}{5 - 4 \cos \theta}$
8. $r = \frac{12}{5 - 5 \cos \theta}$

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9. $\frac{(y + 3)^2}{9} - \frac{(x - 2)^2}{16} = 1$

10. $4(x - 1)^2 - 25(y + 2)^2 = 100$

Use the equation $80x^2 + 120xy + 45y^2 - 100y - 44 = 0$ to complete Exercises 11 and 12.

11. Use the discriminant $B^2 - 4AC$ to identify the graph, and $\tan(2\beta) = \frac{B}{A - C}$ to find $\cos \beta$ and $\sin \beta$.

12. Find the equation in the xy -plane and use a rotation of axes to draw a neat sketch of the graph in the XY -plane.


Graph each polar equation.

13. $r = 3 + 3 \cos \theta$ 14. $r = 4 + 8 \cos \theta$

15. $r = 6 \sin(2\theta)$

For Exercises 16 and 17, identify and graph each conic section from the parametric equations given. Then remove the parameter and convert to rectangular form.

16. $x = 4 \sin t$ 17. $x = (t - 3)^2 + 1$
 $y = 5 \cos t$ $y = t + 2$

 18. Use a graphing calculator to graph the cycloid, then identify the maximum and minimum values, and the period. $x = 4T - 4 \sin T$ $y = 4 - 4 \cos T$

19. Solve each nonlinear system using the technique of your choice.

a. $\begin{cases} 4x^2 - y^2 = 16 \\ y - x = 2 \end{cases}$ b. $\begin{cases} 4y^2 - x^2 = 4 \\ x^2 + y^2 = 4 \end{cases}$

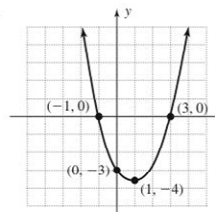
20. Halley's comet has a perihelion of 54.5 million miles and an aphelion of 3253 million miles. Use this information to find a polar equation that models its elliptical orbit. How does its eccentricity compare with that of the planets in our solar system?

21. The soccer match is tied, with time running out. In a desperate attempt to win, the opposing coach pulls his goalie and substitutes a forward. Suddenly, Marques gets a break-away and has an open shot at the empty net, 165 ft away. If the kick is on-line and leaves his foot at an angle of 28° with an initial velocity of 80 ft/sec, is the ball likely to go in the net and score the winning goal?

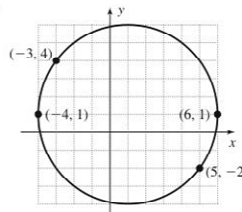
22. The orbit of Mars around the Sun is elliptical, with the Sun at one foci. When the orbit is expressed as a central ellipse on the coordinate grid, its equation is $\frac{x^2}{(141.65)^2} + \frac{y^2}{(141.03)^2} = 1$. Use this information to find the *aphelion* of Mars and the *perihelion* of Mars in millions of miles.

Determine the equation of each relation and state its domain and range. For the parabola and the ellipse, also give the location of the foci.

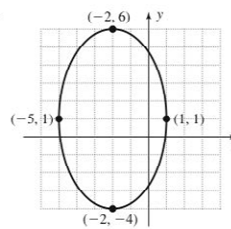
23.



24.



25.



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1114

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10. Analytic Geometry and the Conic Sections

Calculator Exploration and Discovery: Conic Rotations in Polar Form

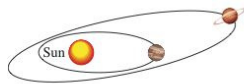
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CALCULATOR EXPLORATION AND DISCOVERY

Conic Rotations in Polar Form

While all planets orbit around the sun in an elliptical path, their **ecliptic planes**, or the planes containing the orbits, differ considerably. For example, using the ecliptic plane of the Earth for refer-



ence, the plane containing Mercury's orbit is inclined by 7° and the plane of the dwarf planet Pluto by 17° ! In addition, if we use the major axis of Earth's orbit for reference, the major axes of the other planets, assuming they are transformed to the ecliptic plane, are rotated by some angle θ . We can gain a basic understanding of the rotations

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10-95

of an elliptical path (relative to some point of reference) using skills developed in this chapter. Here we've seen that the equation of a conic can be given in rectangular form, polar form, and parametric form. Each form seems to have its advantages. When it comes to the rotations of a conic section, it's hard to match the ease and versatility of the polar form. To illustrate, recall that in polar form the general equation of a horizontal ellipse with one focus (the Sun) at the origin is $r = \frac{a(1 - e^2)}{1 - e \cos \theta}$. The constant a gives the length of the semimajor axis and e represents the eccentricity of the orbit. With the exception of Mercury and Pluto (a dwarf planet), the orbits of most planets are close to circular (e is very near zero). This makes the rotations difficult to see. Instead we will explore the concept of axes rotation using "planets" with higher eccentricities. Consider the following planets and their orbital equations. The planet Agnesi has an eccentricity of $e = 0.5$, while the planet Erdős is the most eccentric at $e = 0.75$.

$$\begin{aligned} \text{Agnesi: } & \frac{2.9}{1 - 0.5 \cos \theta} \\ \text{Galois: } & \frac{5.75}{1 - 0.7 \cos \theta} \\ \text{Erdős: } & \frac{7.875}{1 - 0.75 \cos \theta} \end{aligned}$$

We'll investigate the concept of conic rotations in polar form by rotating these ellipses. With your calculator in polar **MODE**, enter these three equations on the **Y=** screen and use the settings shown in Figure 10.77 to set the window size (use $\theta_{\max} = 7$).

```

WINDOW
↑θstep=.1
Xmin=-10
Xmax=32
Xscl=4
Ymin=-15
Ymax=15
Yscl=3
    
```

Figure 10.77

The resulting graph is displayed in Figure 10.78, showing the very hypothetical case where all planets share the same major axis. To show a more realistic case where the planets approach the Sun along orbits with differing major axes, we'll use Galois as a reference and rotate Agnesi $\frac{\pi}{4}$ rad clockwise and Erdős $\frac{\pi}{12}$ rad counter-

clockwise. This is done by simply adjusting the argument of cosine in each equation, using $\cos\left(\theta - \frac{\pi}{4}\right)$ for Agnesi and $\cos\left(\theta + \frac{\pi}{12}\right)$ for

Strengthening Core Skills

1013

Figure 10.78

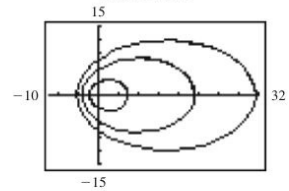


Figure 10.79

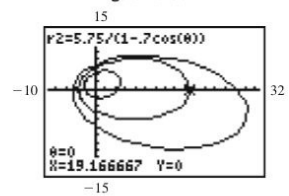
```

Plot1 Plot2 Plot3
Y1=2.9/(1-.5cos
(θ-π/4))
Y2=5.75/(1-.7co
s(θ))
Y3=7.875/(1-.75
cos(θ+π/12))
Y4=
    
```

Erdős. The adjusted **Y=** screen is shown in Figure 10.79, and new graphs in Figure 10.80.

Use these ideas to explore and investigate other rotations by completing the following exercises.

Figure 10.80



Exercise 1: What happens if the angle of rotation is π ? Is the orbit identical if you rotate by $-\pi$?

Exercise 2: If the denominator in the equation is changed to a sum, what effect does it have on the graph?

Exercise 3: If the sign in the numerator is changed, what effect does it have on how the graph is generated?

Exercise 4: After resetting the orbits as originally given, use trial and error to approximate the smallest angle of rotation required for the orbit of Galois to intersect the orbit of Erdős.

Exercise 5: What minimum rotation is required for the orbit of Galois to intersect the orbit of both Agnesi and Erdős?

Exercise 6: What is the minimum rotation required for the orbit of Agnesi to intersect the orbit of Galois?

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1116

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10. Analytic Geometry and the Conic Sections

Strengthening Core Skills: Simplifying and Streamlining Computations for the Rotation of Axes

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STRENGTHENING CORE SKILLS

Simplifying and Streamlining Computations for the Rotation of Axes

While the calculations involved for eliminating the mixed xy -term require a good deal of concentration, there are a few things we can do to simplify the overall process. Basically this involves two things. First, in Figure 10.81 we've organized the process in flowchart form to help you "see" the sequence involved in finding $\cos \beta$ and $\sin \beta$ (for use in the rotation formulas). Second, calculating x^2 , y^2 , and xy (from the equations $x = X \cos \beta - Y \sin \beta$ and $y = X \sin \beta + Y \cos \beta$) as *single terms and apart from their actual substitution* is somewhat less restrictive and seems to help to streamline the algebra.

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Figure 10.82

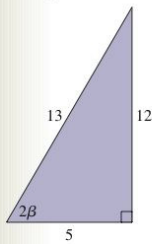
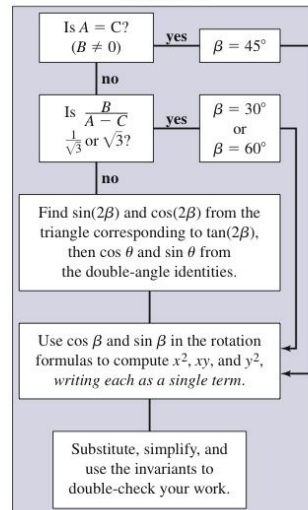


Illustration 1 ▶ For $2x^2 + 12xy - 3y^2 - 42 = 0$, use a rotation of axes to eliminate the xy -term, then identify the conic and its characteristic features.

Solution ▶ Since $A \neq C$, we find β using $\tan(2\beta) = \frac{B}{A - C}$, giving $\tan(2\beta) = \frac{12}{5}$. Using the triangle shown in Figure 10.82 we find $\cos(2\beta) = \frac{5}{13}$. We then find the values of $\cos \beta$ and $\sin \beta$ (choosing 2β in QII), using the double-angle identities as follows:

$$\begin{aligned} \cos \beta &= \sqrt{\frac{1 + \cos(2\beta)}{2}} \\ &= \sqrt{\frac{1 + \frac{5}{13}}{2}} \rightarrow \sqrt{\frac{18}{13}} \\ &= \frac{3}{\sqrt{13}} \\ \sin \beta &= \sqrt{\frac{1 - \cos(2\beta)}{2}} \\ &= \sqrt{\frac{1 - \frac{5}{13}}{2}} \rightarrow \sqrt{\frac{8}{13}} \\ &= \frac{2}{\sqrt{13}} \end{aligned}$$

Figure 10.81



We now compute x^2 , xy , and y^2 prior to substitution in the original equation, writing each as a single term:

$$\begin{aligned} \bullet \quad x &= \frac{3}{\sqrt{13}}X - \frac{2}{\sqrt{13}}Y = \frac{3X - 2Y}{\sqrt{13}} & \bullet \quad y &= \frac{2}{\sqrt{13}}X + \frac{3}{\sqrt{13}}Y = \frac{2X + 3Y}{\sqrt{13}} & \bullet \quad xy &= \frac{(3X - 2Y)(2X + 3Y)}{\sqrt{13}} \\ \bullet \quad x^2 &= \left(\frac{3X - 2Y}{\sqrt{13}}\right)^2 & \bullet \quad y^2 &= \left(\frac{2X + 3Y}{\sqrt{13}}\right)^2 & &= \frac{6X^2 + 5XY - 6Y^2}{13} \\ &= \frac{9X^2 - 12XY + 4Y^2}{13} & &= \frac{4X^2 + 12XY + 9Y^2}{13} & & \end{aligned}$$

Next, we substitute into the original equation, clearing denominators *prior* to using the distributive property.

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1118

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10. Analytic Geometry and
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10-97

Cumulative Review Chapters 1-10

1015

$$42 = 2x^2 + 12xy - 3y^2$$

$$42 = 2\left(\frac{9X^2 - 12XY + 4Y^2}{13}\right) + 12\left(\frac{6X^2 + 5XY - 6Y^2}{13}\right) - 3\left(\frac{4x^2 + 12XY + 9Y^2}{13}\right)$$

multiply both sides by 13, then distribute

$$546 = 18X^2 - 24XY + 8Y^2 + 72X^2 + 60XY - 72Y^2 - 12X^2 - 36XY - 27Y^2$$

$$546 = 78X^2 - 91Y^2$$

combine like terms

$$42 = 6X^2 - 7Y^2$$

simplify and check invariants: $F = f \checkmark$ $A + C = a + c \checkmark$
 $B^2 - 4AC = b^2 - 4ac \checkmark$

$$1 = \frac{X^2}{(\sqrt{7})^2} - \frac{Y^2}{(\sqrt{6})^2}$$

standard form

The graph is a central hyperbola along the X -axis, with vertices at $(\pm\sqrt{7}, 0)$ and asymptotes $Y = \pm\sqrt{\frac{6}{7}}X$.

Exercise 1: Return to Section 10.6 and resolve Exercises 31 and 32 using these methods. Do the new ideas make a difference?

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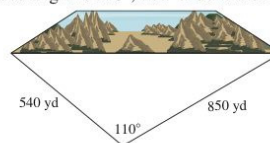


CUMULATIVE REVIEW CHAPTERS 1–10

Solve each equation.

- $\sqrt{x+2} + 2 = \sqrt{3x+4}$
- $x^2 - 6x + 13 = 0$
- $4 \cdot 2^{x+1} = \frac{1}{8}$
- $3^{x-2} = 7$
- $\log_3 81 = x$
- $\log_3 x + \log_3(x-2) = 1$
- $-6 \tan x = 2\sqrt{3}$
- $25 \sin\left(\frac{\pi}{3}x - \frac{\pi}{6}\right) + 3 = 15.5$
- $\frac{\sin 27^\circ}{18} = \frac{\sin x}{35}$
- Use De Moivre's theorem to find the three cube roots of $-8i$. Write the roots in $a + bi$ form.
- The price of beef in Argentina varies directly with demand and inversely with supply. In the small town of Chascomus, the tender-cut lomito was selling for 18 pesos/kg last week. There were 1000 kg available, and 850 kg were bought. Next week there is a 3-day weekend, so the demand is expected to be closer to 1400 kg, but the butchers will only be able to supply 1200 kg. What will a kilogram of tender-cut lomito cost next week?
- Find the inverse of $f(x) = 3 \sin(2x + 1)$.

- A surveyor needs to estimate the width of a large rock formation in Canyonlands National Park. From her current position she is 540 yd from one edge of the formation and 850 yd from the other edge. If the included angle is 110° , how wide is the formation?



Graph each relation. Include vertices, x - and y -intercepts, asymptotes, and other features.

- $f(x) = |x - 2| + 3$
- $y = \sqrt{x - 3} + 1$
- $g(x) = (x - 3)(x + 1)(x + 4)$
- $h(x) = \frac{x - 2}{x^2 - 9}$
- $y = 2^x + 3$
- $f(x) = \log_2(x + 1)$
- $x^2 + y^2 + 10x - 4y + 20 = 0$
- $4(x - 1)^2 - 36(y + 2)^2 = 144$
- $y = -2 \cos\left(x - \frac{\pi}{4}\right) + 1$

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23. $r = 4 \cos(2\theta)$

24. $x = 2 \sin t$
 $y = \tan t$

25. Use the dot product to find the angle between the vectors
- $\mathbf{u} = \langle -4, 5 \rangle$
- and
- $\mathbf{v} = \langle 3, 7 \rangle$
- .

Solve each system of equations.

26.
$$\begin{cases} 4x + 3y = 13 \\ -9y + 5z = 19 \\ x - 4z = -4 \end{cases}$$

27.
$$\begin{cases} x^2 + y^2 = 25 \\ 64x^2 + 12y^2 = 768 \end{cases}$$

28. Find the equation of the parabola with vertex at
- $(2, 3)$
- and directrix
- $x = 0$
- .

29. Decompose
- $y = \frac{3x^3 - 2x^2 + x - 3}{x^4 + x^2}$
- into partial fractions.

30. In the summer, Hollywood releases its big budget, big star, big money movies. Suppose the weekly summer revenue generated by ticket sales was modeled by the function
- $R(w) = -w^4 + 25w^3 - 200w^2 + 560w - 234$
- , where
- $R(w)$
- represents the revenue generated in week
- w
- and
- $1 \leq w \leq 12$
- . Use the remainder theorem to determine the amount of revenue generated in week 5.