WAVELETS FOR KIDS
A Tutorial Introduction

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Strictly speaking, wavelets are topic of pure mathematics, however in only a few years of existence as a theory of their own, they have shown great potential and applicability in many fields.

There are several excellent monographs and articles talking about wavelets, and this modest tutorial does not intend to compete with any of them. Rather it is intended to serve as a very first reading, giving examples interesting for the statistical community. We also give references for further reading as well as some MATHEMATICA do-it-yourself procedures.

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Contents

1 What are wavelets?

Wavelets are functions that satisfy certain requirements. The very name *wavelet* comes from the requirement that they should integrate to zero, "waving" above and below the x-axis. The diminutive connotation of *wavelet* suggest the function has to be well localized. Other requirements are technical and needed mostly to insure quick and easy calculation of the direct and inverse wavelet transform.

There are many kinds of wavelets. One can choose between smooth wavelets, compactly supported wavelets, wavelets with simple mathematical expressions, wavelets with simple associated filters, etc. The most simple is the *Haar wavelet,* and we discuss it as an introductory example in the next section. Examples of some wavelets (from the family of Daubechies wavelets) are given in Figure 1. Like sines and cosines in Fourier analysis, wavelets are used as basis functions in representing other functions. Once the wavelet (sometimes called *the mother wavelet*) $\psi(x)$ is fixed, one can make a basis of translations and dilations of the mother wavelet $\{\psi(\frac{x-b}{a}), (a,b) \in R^+ \times R\}$. It is convenient to take special values for a and b in defining the wavelet basis: $a = 2^{-j}$ and $b = k \cdot 2^j$, where k and j are integers. This choice of a and b will give a sparse basis. In addition, this choice naturally connects multiresolution analysis in signal processing with the world of wavelets.

Wavelet novices often ask, why not use the traditional Fourier methods? There are some important differences between Fourier analysis and wavelets. Fourier basis functions are localized in frequency but not in time. Small frequency changes in the Fourier transform will produce changes everywhere in the time domain. Wavelets are local in both frequency/scale (via dilations) and in time (via translations). This localization is an advantage in many cases.

Second, many classes of functions can be represented by wavelets in a more compact way. For example, functions with discontinuities and functions with sharp spikes usually take substantially fewer wavelet basis functions than sine-cosine basis functions to achieve a comparable approximation.

This sparse coding makes wavelets excellent tools in data compression. For example, the FBI has standardized the use of wavelets in digital fingerprint image compression. The compression ratios are on the order of 20:1, and the difference between the original image and the decompressed one can be told only by an expert. There are many more applications of wavelets, some of them very pleasing. Coifman and his Yale team used wavelets to clean noisy sound recordings, including old recordings of Brahms playing his *First Hungarian Dance* on the piano.

This already hints at how statisticians can benefit from wavelets. Large and noisy data sets can be easily and quickly transformed by the discrete wavelet transform (the counterpart of discrete Fourier transform). The data are coded by the wavelet coefficients. In addition, the epithet "Fast" for Fourier transform can, in most cases, be replaced by "Faster" for the wavelets. It is well known that the computational complexity of the FFT is $O(n \cdot \log_2(n))$. For the fast wavelet transform (FWT) the

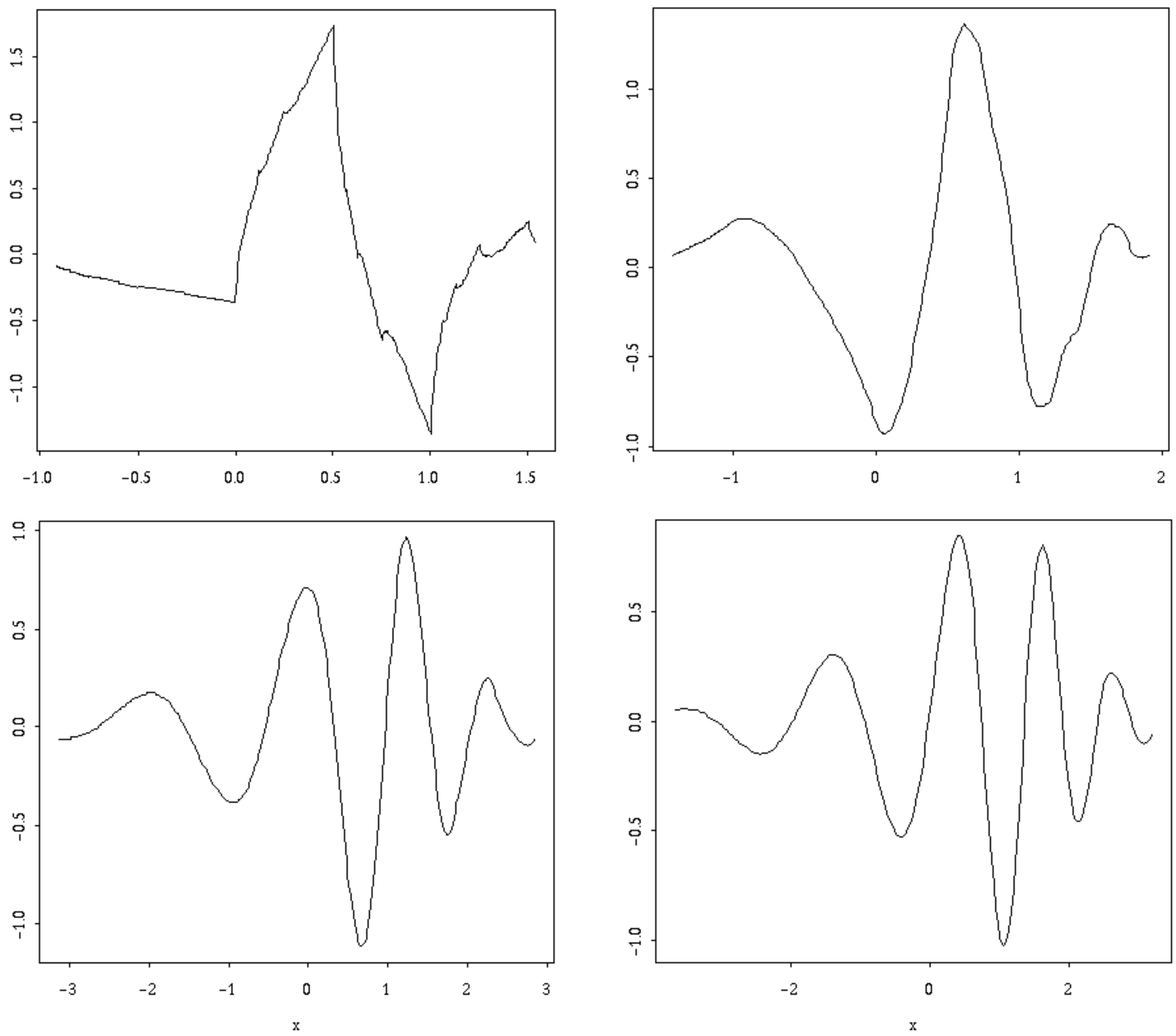


Figure 1: Wavelets from the Daubechies family

computational complexity goes down to $O(n)$.

Many data operations can now be done by processing the corresponding wavelet coefficients. For instance, one can do data smoothing by thresholding the wavelet coefficients and then returning the thresholded code to the "time domain." The definition of thresholding and different thresholding methods are given in Section 3.

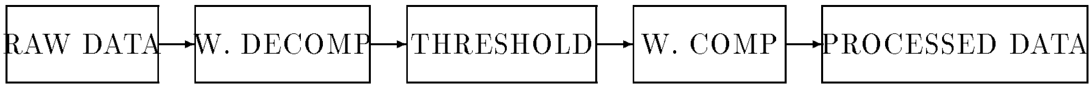


Figure 2: Data analysis by wavelets

2 How do the wavelets work?

2.1 The Haar wavelet

To explain how wavelets work, we start with an example. We choose the simplest and the oldest of all wavelets (we are tempted to say: mother of all wavelets!), the Haar wavelet, $\psi(x)$. It is a step function taking values 1 and -1, on $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$, respectively. The graph of the Haar wavelet is given in Figure 3.

The Haar wavelet has been known for more than eighty years and has been used in various mathematical fields. It is known that any continuous function can be approximated uniformly by Haar functions. (Brownian motion can even be defined by using the Haar wavelet.[1]) Dilations and translations of the function ψ,

$$\psi_{jk}(x) = \text{const} \cdot \psi(2^j x - k),$$

define an orthogonal basis in $L^2(R)$ (the space of all square integrable functions). This means that any element in $L^2(R)$ may be represented as a linear combination (possibly infinite) of these basis functions.

The orthogonality of ψ_{jk} is easy to check. It is apparent that

$$\int \psi_{jk} \cdot \psi_{j'k'} = 0, \tag{1}$$

whenever $j = j'$ and $k = k'$ is not satisfied simultaneously.

If $j \neq j'$ (say $j' < j$), then nonzero values of the wavelet $\psi_{j'k'}$ are contained in the set where the wavelet ψ_{jk} is constant. That makes integral (1) equal to zero:

If $j = j'$, but $k \neq k'$, then at least one factor in the product $\psi_{j'k'} \cdot \psi_{jk}$ is zero. Thus the functions ψ_{ij} are orthogonal.

[1] If $\xi \sim_{iid} N(0,1)$ and $S_{jk}(t) = \int_0^t \psi_{jk}(x)dx$, then $B_t =_{def} \Sigma_{j=1}^{\infty} \Sigma_{k=0}^{2^j-1} \xi_{jk} S_{jk}(t)$ (**P. Levy**).

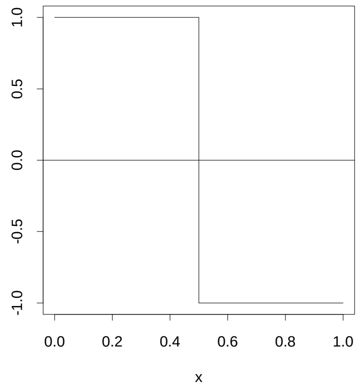


Figure 3: Haar wavelet

The constant that makes this orthogonal basis orthonormal is $2^{j/2}$. Indeed, from the definition of norm[2] in L^2 :

$$1 = (\text{const})^2 \int \psi^2(2^j x - k)dx = (\text{const})^2 \cdot 2^{-j} \int \psi^2(t)dt = (\text{const})^2 \cdot 2^{-j}.$$

The functions $\psi_{10}, \psi_{11}, \psi_{20}, \psi_{21}, \psi_{22}, \psi_{23}$ are depicted in Figure 4. The set $\{\psi_{jk}, j \in Z, k \in Z\}$ defines an orthonormal basis for L^2. Alternatively we will consider orthonormal bases of the form $\{\phi_{j_0,k}, \psi_{jk}, j \geq j_0, k \in Z\}$, where ϕ_{00} is called the *scaling function* associated with the wavelet basis ψ_{jk}. The set $\{\phi_{j_0 k}, k \in Z\}$ spans the same subspace as $\{\psi_{jk}, j < j_0, k \in Z\}$. We will later make this statement more formal and define ϕ_{jk}. For the Haar wavelet basis the scaling function is very simple. It is unity on the interval [0,1), i.e.

$$\phi(x) = \mathbf{1}(0 \leq x < 1).$$

The statistician may be interested in wavelet representations of functions generated by data sets.

Let $\underset{\sim}{y} = (y_0, y_1, \ldots, y_{2^n-1})$ be the data vector of size 2^n. The data vector can be associated with a piecewise constant function f on [0,1) generated by $\underset{\sim}{y}$ as follows,

$$f(x) = \Sigma_{k=0}^{2^n-1} \; y_k \; \cdot \; \mathbf{1}(k2^{-n} \leq x < (k+1)2^{-n}).$$

The (data) function f is obviously in the $L^2[0,1)$ space, and the wavelet decomposition of f has the form

$$f(x) = c_{00}\phi(x) + \Sigma_{j=0}^{n-1}\Sigma_{k=0}^{2^j-1} d_{jk}\psi_{jk}(x). \tag{2}$$

The sum with respect to j is finite because f is a step function, and everything can be exactly described by resolutions up to the $(n-1)$-st level. For each level the sum

[2] $||f||^2 =_{def} \langle f, f\rangle = \int f^2$.

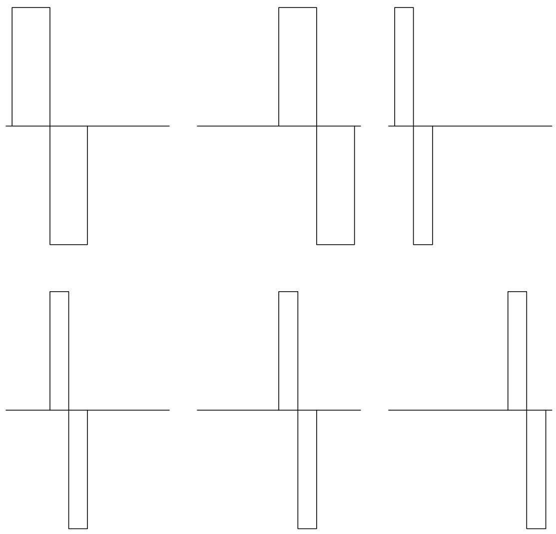


Figure 4: Dilations and translations of Haar wavelet on [0,1]

with respect to k is also finite because the domain of f is finite. In particular, no translations of the scaling function ϕ_{00} are required.

We fix the data vector y and find the wavelet decomposition (2) explicitly. Let $\tilde{y} = (1, 0, -3, 2, 1, 0, 1, 2)$. The corresponding function f is given in Figure 5. The following matrix equation gives the connection between $\tilde{y}$ and the wavelet coefficients. Note the constants 2^j ($1, \sqrt{2}$ and 2) with Haar wavelets on the corresponding resolution levels (j=0, 1, and 2).

$$\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \\ 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \sqrt{2} & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & \sqrt{2} & 0 & -2 & 0 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & -2 & 0 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & -2 & 0 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & 2 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} c_{00} \\ d_{00} \\ d_{10} \\ d_{11} \\ d_{20} \\ d_{21} \\ d_{22} \\ d_{23} \end{bmatrix}$$

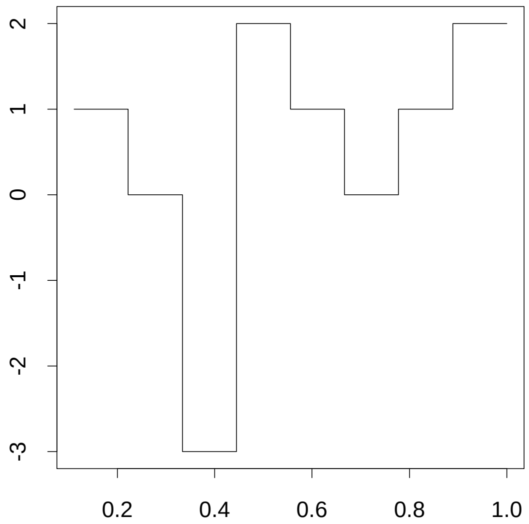


Figure 5: "Data function" on [0,1)

The solution is

$$\begin{bmatrix} c_{00} \\ d_{00} \\ d_{10} \\ d_{11} \\ d_{20} \\ d_{21} \\ d_{22} \\ d_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \\ \frac{1}{4} \\ -\frac{5}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}.$$

Thus,

$$f = \frac{1}{2}\phi - \frac{1}{2}\psi_{00} + \frac{1}{2\sqrt{2}}\psi_{10} - \frac{1}{2\sqrt{2}}\psi_{11} + \frac{1}{4}\psi_{20} - \frac{5}{4}\psi_{21} + \frac{1}{4}\psi_{22} - \frac{1}{4}\psi_{23} \tag{3}$$

The solution is easy to check. For example, when $x \in [0, \frac{1}{8})$,

$$f(x) = \frac{1}{2} - \frac{1}{2} \cdot 1 + \frac{1}{2\sqrt{2}} \cdot \sqrt{2} + \frac{1}{4} \cdot 2 = 1.$$

The reader may already have the following question ready: "What will we do for vectors $\tilde{y}$ of much bigger length?" Obviously, solving the matrix equations becomes impossible.

2.2 Mallat's multiresolution analysis, filters, and direct and inverse wavelet transformation

An obvious disadvantage of the Haar wavelet is that it is not continuous, and therefore choice of the Haar basis for representing smooth functions, for example, is not natural and economic.

2.2.1 Mallat's MRA

As a more general framework we explain Mallat's Multiresolution Analysis – (MRA). The MRA is a tool for a constructive description of different wavelet bases.

We start with the space L^2 of all square integrable functions.[3] The MRA is an increasing sequence of closed subspaces $\{V_j\}_{j\in Z}$ which approximate $L^2(R)$.

Everything starts with a clever choice of the *scaling function* ϕ. Except for the Haar wavelet basis for which ϕ is the characteristic function of the interval $[0, 1)$, the scaling function is chosen to satisfy some continuity, smoothness and tail requirements. But, most importantly, the family $\{\phi(x-k), \quad k \in Z\}$ forms an orthonormal basis for the *reference* space V_0. The following relations describe the analysis.

MRA 1 $\qquad \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$

The spaces V_j are nested. The space $L^2(R)$ is a closure of the union of all V_j. In other words, $\cup_{j\in Z} V_j$ is dense in $L^2(R)$. The intersection of all V_j is empty.

MRA 2 $\qquad f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1},\ j \in Z.$

The spaces V_j and V_{j+1} are "similar." If the space V_j is spanned by $\phi_{jk}(x), k \in Z$ then the space V_{j+1} is spanned by $\phi_{j+1,k}(x), k \in Z$. The space V_{j+1} is generated by the functions $\phi_{j+1,k}(x) = \sqrt{2}\phi_{jk}(2x)$.

We now explain how the wavelets enter the picture. Because $V_0 \subset V_1$, any function in V_0 can be written as a linear combination of the basis functions $\sqrt{2}\phi(2x-k)$ from V_1. In particular:

$$\phi(x) = \Sigma_k h(k)\sqrt{2}\phi(2x-k). \tag{4}$$

Coefficients $h(k)$ are defined as $\langle \phi(x), \sqrt{2}\phi(2x-k)\rangle$. Consider now the orthogonal complement W_j of V_j to V_{j+1} (i.e. $V_{j+1} = V_j \oplus W_j$). Define

$$\psi(x) = \sqrt{2}\Sigma_k(-1)^k h(-k+1)\phi(2x-k). \tag{5}$$

It can be shown that $\{\sqrt{2}\psi(2x-k), k \in Z\}$ is an orthonormal basis for W_1.[4]

[3] A function f is in $L^2(S)$ if $\int_S f^2$ is finite.

[4] This can also be expressed in terms of Fourier transformations as follows: Let $m_0(\omega)$ be the

Again, the similarity property of MRA gives that $\{2^{j/2}\psi(2^j x - k), k \in Z\}$ is a basis for W_j. Since $\cup_{j\in Z}V_j = \cup_{j\in Z}W_j$ is dense in $L_2(R)$, the family $\{\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k), j \in Z, k \in Z\}$ is a basis for $L^2(R)$.

For a given function $f \in L^2(R)$ one can find N such that $f_N \in V_N$ approximates f up to preassigned precision (in terms of L_2 closeness). If $g_i \in W_i$ and $f_i \in V_i$, then

$$f_N = f_{N-1} + g_{N-1} = \Sigma_{i=1}^{M} g_{N-M} + f_{N-M}. \tag{6}$$

Equation (6) is the wavelet decomposition of f. For example, the data function (2.1) is in V_n, if we use the MRA corresponding to the Haar wavelet. Note that $f \equiv f_n$ and $f_0 = 0$.

2.2.2 The language of signal processing

We repeat the multiresolution analysis story in the language of signal processing theory. Mallat's multiresolution analysis is connected with so called "pyramidal" algorithms in signal processing. Also, "quadrature mirror filters" are hidden in Mallat's MRA.

Recall from the previous section that

$$\phi(x) = \Sigma_{k\in Z} h(k)\sqrt{2}\phi(2x - k), \tag{7}$$

and

$$\psi(x) = \Sigma_{k\in Z} g(k)\sqrt{2}\phi(2x - k). \tag{8}$$

The l^2 sequences[5] $\{h(k), k \in Z\}$ and $\{g(k), k \in Z\}$ are *quadrature mirror filters* in the terminology of signal analysis. The connection between h and g is given by:

$$g(n) = (-1)^n h(1 - n).$$

The sequence $h(k)$ is known as a *low pass* or *low band* filter while $g(k)$ is known as the *high pass* or *high band* filter. The following properties of $h(n), g(n)$ can be proven by using Fourier transforms and orthogonality: $\Sigma h(k) = \sqrt{2}, \quad \Sigma g(k) = 0$.

The most compact way to describe the Mallat's MRA as well to give effective procedures of determining the wavelet coefficients is the *operator representation of filters.*

Fourier transformation of the sequence $h(n), n \in Z$, i.e. $m_0(\omega) = \Sigma_n h(n)e^{in\omega}$. In the 'frequency domain" the relation (4) is $\hat{\phi}(\omega) = m_0(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2})$. If we define $m_1(\omega) = e^{-i\omega}m_0(\omega + \pi)$ and $\hat{\psi}(2\omega) = m_1(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2})$, then the function ψ corresponding to $\hat{\psi}$ is *the wavelet associated with the* MRA.

[5] A sequence $\{a_n\}$ is in the Hilbert space l^2 if $\Sigma_{k\in Z} a_k^2$ is finite.

For a sequence $a = \{a_n\}$ the operators H and G are defined by the following coordinatewise relations:

$$\begin{aligned}(Ha)_k &= \Sigma_n h(n-2k)a_n \\ (Ga)_k &= \Sigma_n g(n-2k)a_n.\end{aligned}$$

The operators H and G correspond to one step in the wavelet decomposition. The only difference is that the above definitions do not include the $\sqrt{2}$ factor as in Equations (4) and (5).

Denote the original signal by $\underset{\sim}{c}^{(n)}$. If the signal is of length 2^n, then $\underset{\sim}{c}^{(n)}$ can be represented by the function $f(x) = \Sigma \underset{\sim}{c}_k^{(n)} \phi_{nk}$, $f \in V_n$. At each stage of the wavelet transformation we move to a coarser approximation $\underset{\sim}{c}^{(j-1)}$ by $\underset{\sim}{c}^{(j-1)} = H\underset{\sim}{c}^{(j)}$ and $\underset{\sim}{d}^{(j-1)} = G\underset{\sim}{c}^{(j)}$. Here, $\underset{\sim}{d}^{(j-1)}$ is the "detail" lost by approximating $\underset{\sim}{c}^{(j)}$ by the averaged $\underset{\sim}{c}^{(j-1)}$. The discrete wavelet transformation of a sequence $\underset{\sim}{y} = \underset{\sim}{c}^{(n)}$ of length 2^n can then be represented as another sequence of length 2^n (notice that the sequence $\underset{\sim}{c}^{(j-1)}$ has half the length of $\underset{\sim}{c}^{(j)}$):

$$(\underset{\sim}{d}^{(n-1)}, \underset{\sim}{d}^{(n-2)}, \ldots, \underset{\sim}{d}^{(1)}, \underset{\sim}{d}^{(0)}, \underset{\sim}{c}^{(0)}). \tag{9}$$

Thus the discrete wavelet transformation can be summarized as a single line:

$$\underset{\sim}{y} \longrightarrow (G\underset{\sim}{y}, GH\underset{\sim}{y}, GH^2\underset{\sim}{y}, \ldots, GH^{n-1}\underset{\sim}{y}, H^n\underset{\sim}{y}).$$

The reconstruction formula is also simple in terms of H and G; we first define adjoint operators $H^\star$ and $G^\star$ as follows:

$$\begin{aligned}(H^\star a)_n &= \Sigma_k h(n-2k)a_n \\ (G^\star a)_n &= \Sigma_k g(n-2k)a_n.\end{aligned}$$

Recursive application leads to:

$$(G\underset{\sim}{y}, GH\underset{\sim}{y}, GH^2\underset{\sim}{y}, \ldots, GH^{j-1}\underset{\sim}{y}, H^j\underset{\sim}{y}) \longrightarrow \underset{\sim}{y} = \Sigma_{j=0}^{n-1}(H^\star)^j G^\star \underset{\sim}{d}^{(j)} + (H^\star)^n \underset{\sim}{c}^{(0)}.$$

Equations (7) and (8) which generate filter coefficients (sometimes called *dilation equations*) look very simple for the Haar wavelet:

$$\begin{aligned}\phi(x) &= \phi(2x) + \phi(2x-1) = \frac{1}{\sqrt{2}}\sqrt{2}\phi(2x) + \frac{1}{\sqrt{2}}\sqrt{2}\phi(2x-1), \qquad (10) \\ \psi(x) &= \phi(2x) - \phi(2x-1) = \frac{1}{\sqrt{2}}\sqrt{2}\phi(2x) - \frac{1}{\sqrt{2}}\sqrt{2}\phi(2x-1).\end{aligned}$$

The filter coefficients in (10) are

$$h(0) = h(1) = \frac{1}{\sqrt{2}} \qquad g(0) = -g(1) = \frac{1}{\sqrt{2}}$$

$$\underset{\sim}{y} = c^{(3)} \quad \boxed{\begin{array}{cccccccc} 1 & 0 & -3 & 2 & 1 & 0 & 1 & 2 \end{array}}$$

$$\underset{\sim}{d}^{(2)} \quad \boxed{\begin{array}{cccc} \frac{1}{\sqrt{2}} & -\frac{5}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{array}}$$

$$\underset{\sim}{c}^{(2)} \quad \boxed{\begin{array}{cccc} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{array}}$$

$$\underset{\sim}{d}^{(1)} \quad \boxed{\begin{array}{cc} 1 & -1 \end{array}}$$

$$\underset{\sim}{c}^{(1)} \quad \boxed{\begin{array}{cc} 0 & 2 \end{array}}$$

$$\underset{\sim}{d}^{(0)} \quad \boxed{-\sqrt{2}}$$

$$\underset{\sim}{c}^{(0)} \quad \boxed{\sqrt{2}}$$

Figure 6: Decomposition procedure

Figure 6 schematically gives the decomposition algorithm applied to our data set.

To get the wavelet coefficients as in (3) we multiply components of $\underset{\sim}{d}^{(j)},\ j = 0, 1, 2$ and $c^{(0)}$ with the factor $2^{-N/2}$. Simply,

$$d_{jk} = 2^{-N/2} d_k^{(j)}, \quad 0 \le j < N \ (= 3).$$

It is interesting that in the Haar wavelet case $2^{-3/2} c_0^{(0)} = c_{00} = \frac{1}{2}$ is the mean of the sample $\underset{\sim}{y}$.

Figure 7 schematically gives the reconstruction algorithm for our example.

The careful reader might have already noticed that when the length of the filter is larger than 2, boundary problems occur. (There are no boundary problems with the Haar wavelet!) There are two main ways to handle the boundaries: *symmetric* and *periodic*.

We should remark that some problems call for use of *continuous wavelet transforms.*

3 Thresholding methods

In wavelet decomposition the filter H is an "averaging" filter while its mirror counterpart G produces details. The wavelet coefficients correspond to details. When details are small, they might be omitted without substantially affecting the "general picture." Thus the idea of thresholding wavelet coefficients is a way of cleaning out "unimportant" details considered to be noise. We illustrate the idea on our old friend, the data vector $(1, 0, -3, 2, 1, 0, 1, 2)$.

Example: The data vector $(1, 0, -3, 2, 1, 0, 1, 2)$ is transformed into the vector

$$(\frac{1}{\sqrt{2}}, -\frac{5}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1, -1, -\sqrt{2}, \sqrt{2}).$$

If all coefficients less than 0.9 (well, our choice) are replaced by zeroes, then the resulting ("thresholded") vector is $(0, -\frac{5}{\sqrt{2}}, 0, 0, 1, -1, -\sqrt{2}, \sqrt{2})$.

The graph of "smoothed data", after reconstruction, is given in Figure 8.

Wavelet thresholding has important applications in statistics. Donoho and Johnstone (1993) discuss an application to a nonparametric regression problem. Assume we observe some unknown function f with Gaussian noise: $y_i = f(t_i) + \sigma\epsilon_i$, $i = 1, \ldots, n$ and $\epsilon_i \sim N(0, 1)$. The goal is to estimate the unknown function f.

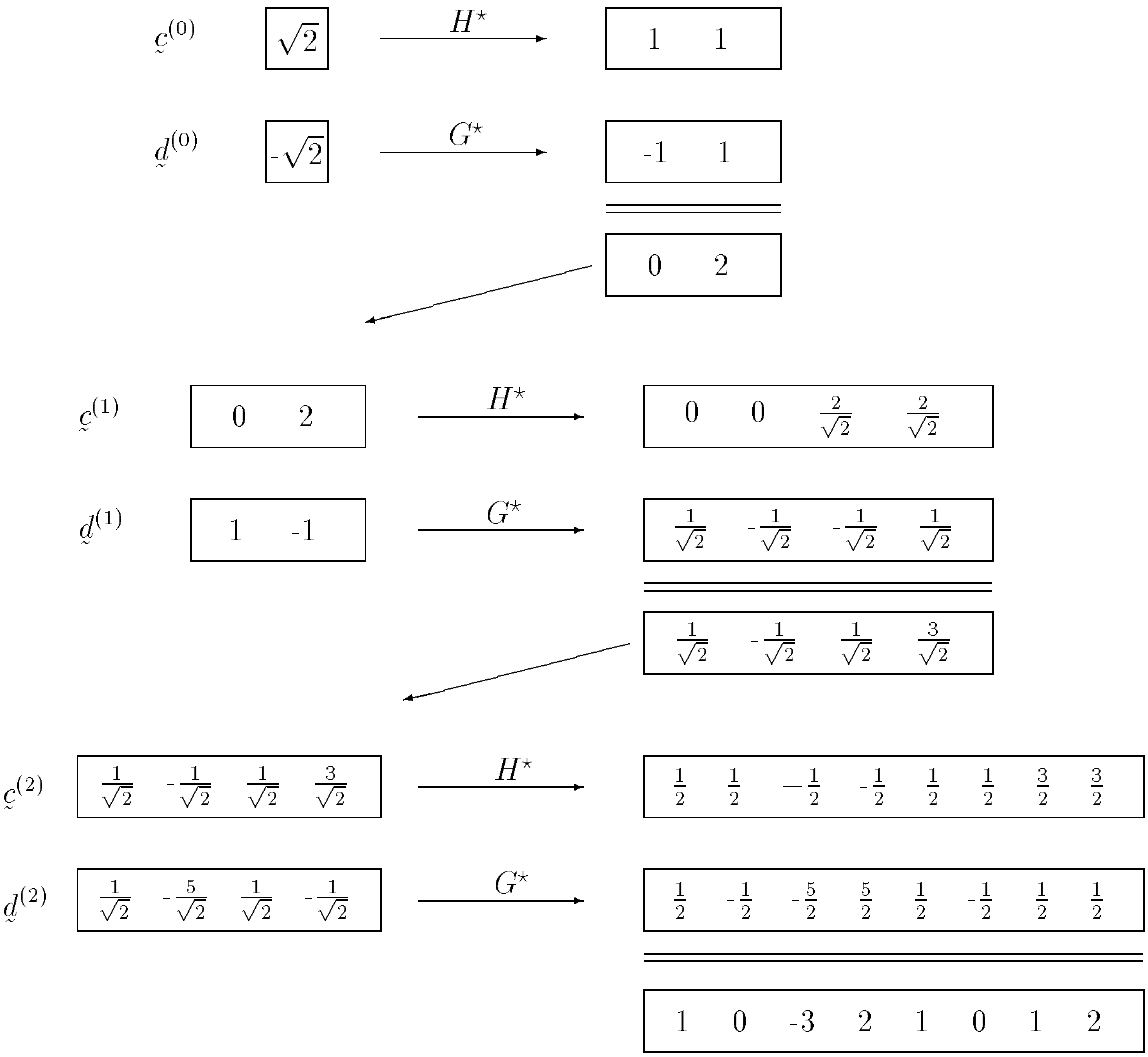


Figure 7: Reconstruction example

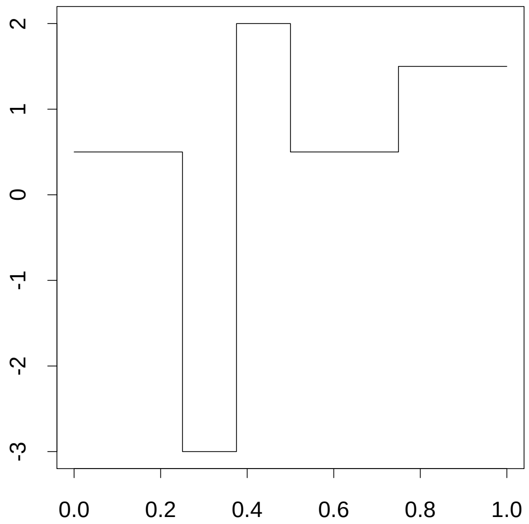


Figure 8: "Smoothed" sequence

Donoho and Johnstone propose to start with a wavelet decomposition of the data set, threshold the coefficients, and then use the wavelet reconstruction as an estimate $\hat{f}$.

When using the thresholding rule $\hat{d}_{jk} = \text{sign}(d_{jk})(|d_{jk}|-\lambda)_+$ with $\lambda = \sigma\sqrt{2\log n}/\sqrt{n}$, the estimate $\hat{f}$ can be shown to have risk $R(\hat{f}, f)$ within a factor $2\log n$ of the minimum risk when using the (of course unknown) optimal thresholding rule. Here R is given by $R(\hat{f}, f) = E(\Sigma(\hat{f}(t_i) - f(t_i))^2/n$. Donoho and Johnstone (1993) show that the (interpolated) function estimate $\hat{f}$ is, with probability tending to 1 (as $n \to \infty$), at least as smooth as f.

Another interesting application of wavelet thresholding arises in density estimation. Assume $X_1, \ldots, X_n$ are i.i.d. observations from an unknown probability density function $f(x)$. Donoho, Johnstone, Kerkyacharian, and Picard (1993) define a non-linear density estimate by thresholding the coefficient in the wavelet decomposition of the empirical p.d.f.

If the unknown density is estimated by

$$\hat{f}(x) = \Sigma_{jk}\hat{d}_{jk}\psi_{jk}(x),$$

then due to orthonormality of ψ_{jk}s, the sample estimator of d_{jk} is $\hat{d}_{jk} = \frac{1}{n}\Sigma\psi_{jk}(X_i)$.

Thresholding in this problem reminds us of well known procedures in density estimation by orthogonal series: shrinking and tapering.

In the next subsection we will give a brief tour through some thresholding policies.

3.1 Different thresholding policies

3.1.1 Hard thresholding

The policy for hard thresholding is *keep or kill.* The absolute values of all wavelet coefficients are compared to a fixed threshold λ. If the magnitude of the coefficient is less than λ, the coefficient is replaced by zero:

$$d_{jk}^{hard} = \begin{cases} 0, d_{jk} < \lambda \\ d_{jk}, d_{jk} \geq \lambda \end{cases}$$

The function performing hard thresholding is given in Figure 9 a.

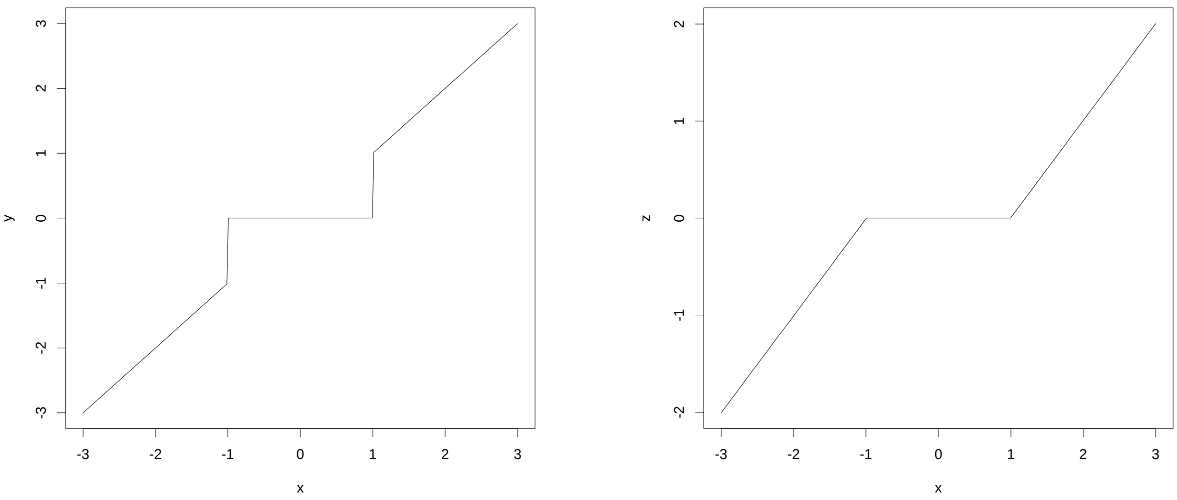


Figure 9: Hard and soft thresholding with $\lambda = 1$.

Hard thresholding is used when one is interested in the shortest possible wavelet code. Long sequences of zeroes that are usually obtained in thresholded wavelet decomposition vector are coded in an efficient way.

3.1.2 Soft thresholding

Soft thresholding shrinks all the coefficients towards the origin. The formula is

$$d_{jk}^{soft} = sign(d_{jk})(|d_{jk}| - \lambda)_+. \tag{11}$$

The graph of the function performing the soft thresholding is given in Figure 9 b.

3.1.3 Quantile thresholding

Let the rule for thresholding be given as

$$d_{jk}^{quant} = \begin{cases} 0, d_{jk} < p \\ d_{jk}, d_{jk} \geq p \end{cases},$$

where p is a p-quantile of the set of all wavelet coefficients. For example, we might want to replace 30% of the smallest wavelet coefficients by zero.

3.1.4 Universal thresholding

Donoho and Johnstone (1992) propose to use the threshold $\lambda = \sigma\sqrt{2\log(n)}/\sqrt{n}$ on transformed data set $\frac{y_i}{n}$, where n is the sample size, and σ is the scale of the noise on a standard deviation scale. Universal thresholding can be hard or soft thresholding with the above defined λ as threshold.

4 Example: California earthquakes

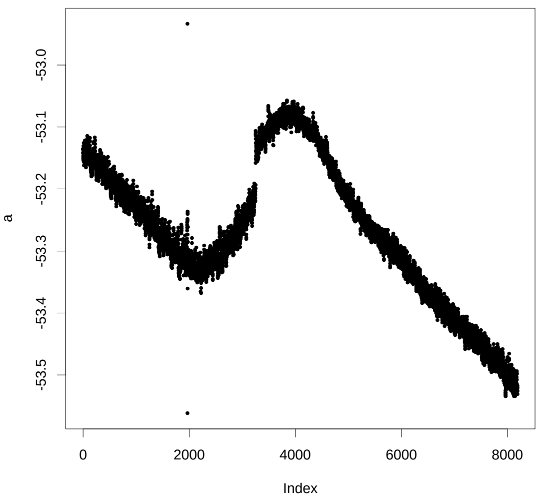
A researcher in geology was interested in predicting earthquakes by the level of water in nearby wells. She had a large ($8192 = 2^{13}$ measurements) data set of water levels taken every hour in a period of time of about one year in a California well. Here is the description of the problem.

> The ability of water wells to act as strain meters has been observed for centuries. The Chinese, for example, have records of water flowing from wells prior to earthquakes. Lab studies indicate that a seismic slip occurs along a fault prior to rupture. Recent work has attempted to quantify this response, in an effort to use water wells as sensitive indicators of volumetric strain. If this is possible, water wells could aid in earthquake prediction by sensing precursory earthquake strain.
>
> We have water level records from six wells in southern California, collected over a six year time span. At least 13 moderate size earthquakes (Magnitude 4.0 - 6.0) occurred in close proximity to the wells during this time interval. There is a a significant amount of noise in the water level record which must first be filtered out. Environmental factors such as earth tides and atmospheric pressure create noise with frequencies ranging from seasonal to semidiurnal. The amount of rainfall also affects the water level, as do surface loading, pumping, recharge (such as an increase in water level due to irrigation), and sonic booms, to name a few. Once the noise is subtracted from the signal, the record can be analyzed for changes in water level, either an increase or a decrease depending upon whether the aquifer is experiencing a tensile or compressional volume strain, just prior to an earthquake.

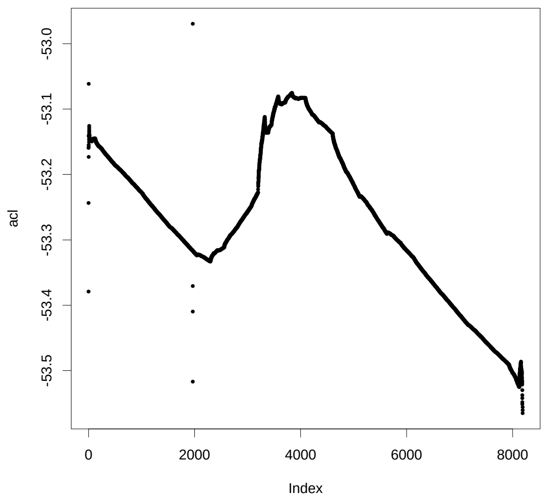
A plot of the raw data for hourly measurements over one year ($8192 = 2^{13}$ observations) is given in Figure 10a. After applying the DAUB #2 wavelet transformation and thresholding by the Donoho-Johnstone "universal" method, we got a very clear signal with big jumps at the earthquake time. The cleaned data are given in Figure 10b. The magnitude of the water level change at the earthquake time did not get distorted in contrast to usual smoothing techniques. This is a desirable feature of wavelet methods. Yet, a couple of things should be addressed with more care.

(i) Possible fluctuations important for the earthquake prediction are cleaned as noise. In post-analyzing the data, having information about the earthquake time, one might do time-sensitive thresholding.

(ii) Small spikes on the smoothed signal (Figure 10b) as well as 'boundary distortions" indicate that the DAUB2 wavelet is not the most fortunate choice. Compromising between smoothness and the support shortness of the mother wavelet with help of wavelet banks, one can develop ad-hoc rules for better mother wavelet (wavelet model) choice.



(a) Raw data, water level vs. time



(b) After thresholding the wavelet transformation.

Figure 10: Panel (a) shows $n = 8192$ hourly measurements of the waterlevel for a well in an earthquake zone. Notice the wide range of waterlevels at the time of an earthquake around $t = 2000$.

5 Wavelet image processing

We will explain briefly how wavelets may be useful in the matrix data processing. The most remarkable application is, without any doubt, image processing. Any (black and white) image can be approximated by a matrix A in which the entries a_{ij} correspond to intensities of gray in the pixel (i, j). For reasons that will be obvious later, it is assumed that A is the square matrix of dimension $2^n \times 2^n$, n integer.

The process of the image wavelet decomposition goes as follows. On the rows of the matrix A the filters H and G are applied. Two resulting matrices are obtained: $H_r A$ and $G_r A$, both of dimension $2^n \times 2^{n-1}$ (Subscript r suggest that the filters are applied on rows of the matrix A). Now on the columns of matrices $H_r A$ and $G_r A$, filters H and G are applied again and the four resulting matrices $H_c H_r A, G_c H_r A, H_c G_r A$ and $G_c G_r A$ of dimension $2^{n-1} \times 2^{n-1}$ are obtained. The matrix $H_c H_r A$ is the average, while the matrices $G_c H_r A, H_c G_r A$ and $G_c G_r A$ are details (Figure 11)

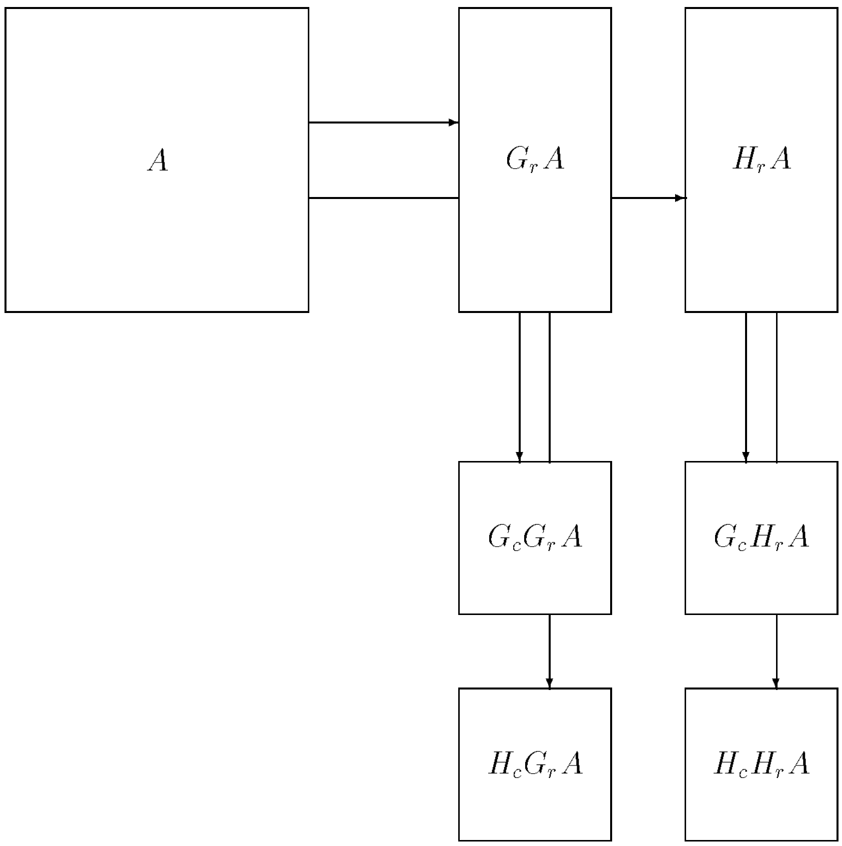


Figure 11: Image wavelet decomposition

The process continues with the *average* matrix $H_c H_r A$ until a single number (*average* of the whole original matrix A) is obtained. Two examples are given below.

Example 1.

This example is borrowed from Nason and Silverman (1993). The top left panel in Figure 12 is 256 $\times$ 256 black and white image of John Lennon in 0-255 gray scale.

In the top-right figure each pixel is contaminated by normal $N(0, 60)$ noise. (In *Splus*: `le` $\leftarrow$ `lennon+rnorm(256*256, s=60)` where `lennon` is the pixel matrix of the original image.)

The two bottom figures are restored images. The DAUB #4 filter was used for the first figure, while DAUB #10 was used for the second.

Though the quality of the restored images may be criticized, the stunning property of wavelet image analysis shows up in this example. Both restored images use only about 1.8 % of the information contained in the "blurred" image. The compression rate is amazing: 527120 bites go to 9695 bites after the universal thresholding.

Example 2.

This is an adaptation of the data set of J. Schmert, University of Washington. The word **five** was recorded and each column on the top-right figure represents a periodogram over a short period of time (adjacent columns have half of their observations in common). The rows represent time. The original 92 $\times$ 64 matrix was cut to 64 $\times$ 64 matrix for obvious reasons. After performing hard thresholding with $\lambda = 0.25$, a compression ratio of 1:2 is achieved. The compressed figures are shown in the two bottom panels of Figure 13.

6 Can you do wavelets?

Yes, you can! There are several several packages that support wavelet calculations. The best (noncommercial) package in our opinion is Nason and Silverman's: *The Discrete Wavelet Transform in S.* The manual [14] describes installation and use of the software is installed and used. The software itself can be **ftp**ed[6] from `lib.stat.cmu.edu` or `hensa.unix.ac.uk`. The name of the package is `wavethresh`.

Carl Taswell (`taswell@sccm.stanford.edu`) developed Wavelet Toolbox for MATHLAB. The latest version is WavBox 4.0 and the software has to be registered with the author.

There are several MATHEMATICA notebooks on wavelet computations. Wickerhauser, Cohen, (`jkc@keller.mines.colorado.edu`), made theirs available to the public.

To understand how the wavelets work, we reinvented the wheel and developed MATHEMATICA software for direct and inverse wavelet transformation and thresholding and applied it to some exemplary data sets. The algorithms are far from

[6]A new verb, ha!

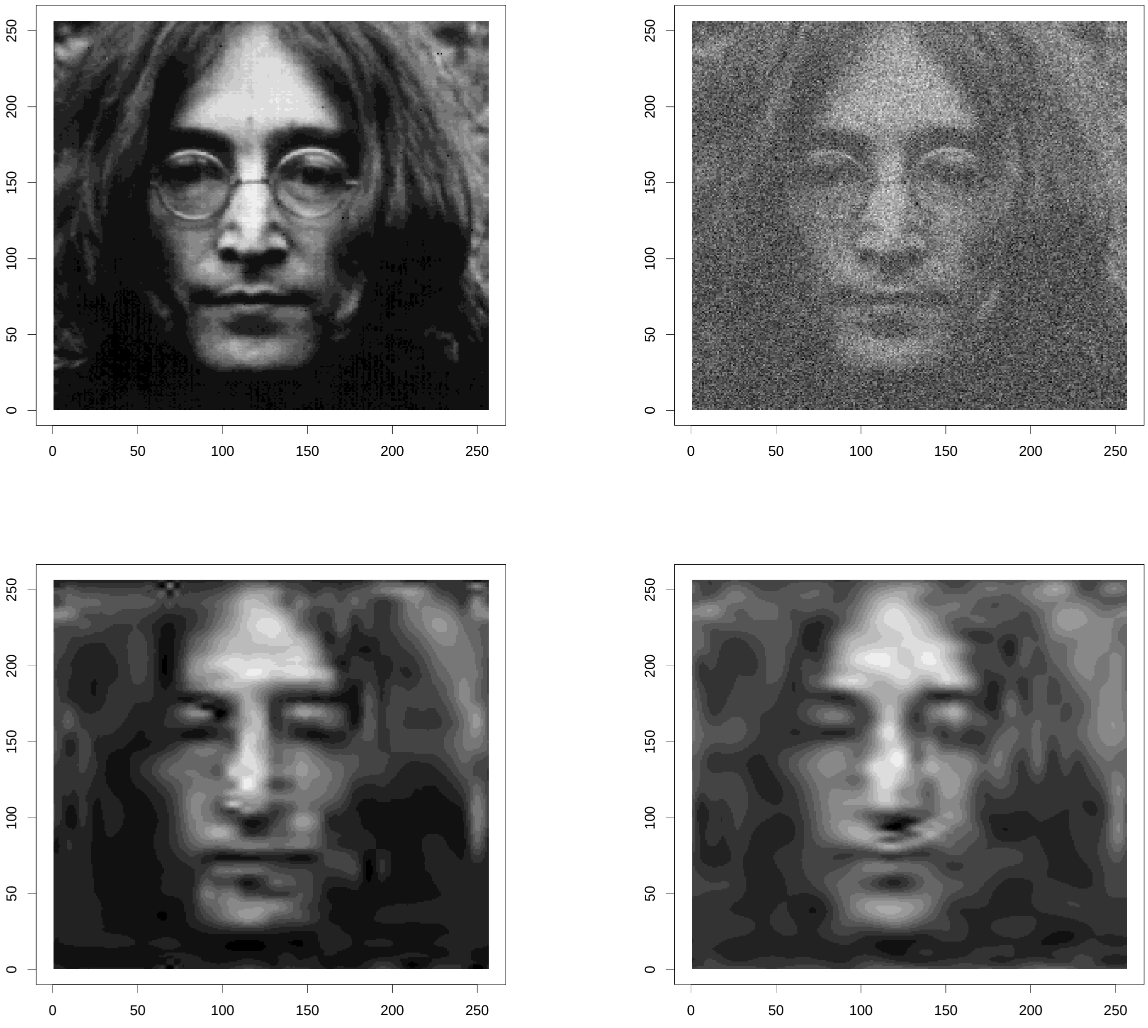


Figure 12: Wavelet image restoration example

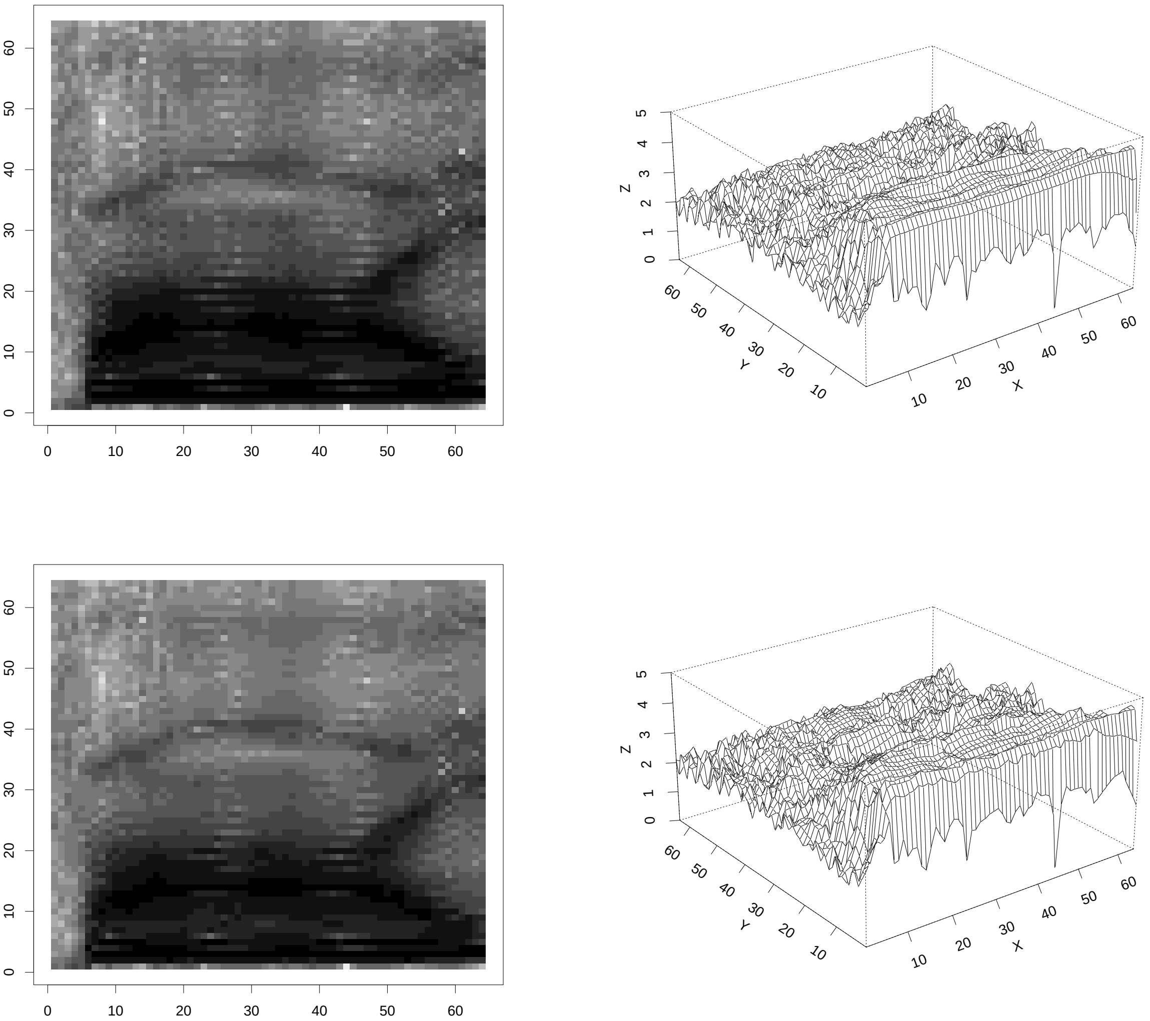


Figure 13: Word **FIVE** data. The panels in the first row show to the original data. The bottom panels show the signal after thresholding.

being effective; rather they are educational. MATHEMATICA notebook is given in the appendix.

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7 Appendix

```
BeginPackage["Waves`"]
(* Author: Brani Vidakovic, ISDS, Duke University ;
Functions Dec and Comp are based on M. V. Wickerhauser's
mathematica program *)

Mirror::usage = "Mirror[_filter_] gives the mirror  \
filter for the input _filter_. This is an adjoint  \
operator H* of the operator H corresponding to  _filter_."

WT::usage = "WT[_vector_,  _filter_]  performs the direct \
 wavelet transformation of the data vector  _vector_. \
The  wavelet  base is chosen by _filter_. The length \
of the vector  _vector_  has to be a degree of 2."

WR::usage = "WR[_vector_,  _filter_] gives the wavelet \
reconstruction algorithm. From the set of  wavelet \
coefficients _vector_ the data set is reconstructed. \
The wavelet base is chosen by _filter_."
```

```
Dec::usage = "An auxiliary function needed for the \
direct wavelet transformation. See WT."

Comp::usage = "An auxiliary function needed for the \
inverse wavelet transformation (wavelet reconstruction  \
algorithm). See WR."

 Begin["`Private`"]


Mirror[ filter_List]:= Module[{fl=Length[filter]},
Table[ -(-1)^i filter[[fl+1-i]], {i, 1, fl}]];


Dec[ vector_List, filter_List]:=  Module[
 {vl= Length[vector], fl=Length[filter]},
 Table[
      Sum[ filter[[m]] vector[[Mod[2 k+m - 3, vl]+1 ]],
         {m,1,fl}],
      {k,1,vl/2}]
 ];


Comp[ vector_List, filter_List]:= Module[
 { temp=Table[0,{i,1,2 Length[vector]}],
 vl=Length[vector], fl=Length[filter]},
 Do[ temp[[ Mod[2 j + i -3, 2 vl]+1]] +=
        vector[[j]] filter[[i]],
      {j, 1, vl}, {i, 1, fl}];
  temp];


WT[ vector_List, filter_List]:=
Module[ { wav={}, c,d, ve=vector, H=filter,
G=Mirror[filter]},
While[ Length[ve] > 1,
     lev=Log[2,Length[ve]]-1;
     c = Dec[ve, H];
     d = Dec[ve, G];
     wav= Join[ wav, d ];
     ve = c]; Join[wav, c] ];
```

```
WR[ vector_List, filter_List]:=
Module[ {i=1, vl=Length[vector], c=Take[vector,-1],
        d=Take[RotateRight[vector,1],-1],
        mirrorf=Mirror[filter], cn, dn, k=1},
While[ i <= vl/2 ,
       k += i;
       i= 2 i;
       cn=Comp[c, filter]+Comp[d, mirrorf];
       dn=Take[RotateRight[vector, k], -i ];
       c=cn;
       d=dn;
     ];
c  ];

End[ ]

EndPackage[ ]
```

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